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ELEMENTARY

ALGEBRA:

FOR

THE USE OF SCHOOLS.

BY WILLIAM SMYTH, A. M.,

PROFESSOR OF MATHEMATICS IN BOWDOIN COLLEGE

THIRD EDITION.

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PREFACE.

The design of the author, in this treatise, is to provide a convenient introduction to his larger work on Algebra. The work, however, has been extended so as to make a whole in itself, adapted to the use of a large class of pupils desirous of acquiring some general knowledge of Algebra, but who have comparatively little time to devote to it. The following analysis will exhibit the general plan of the author:

In the first section, the algebraic signs are explained, and the fundamental idea impressed upon the learner that these signs constitute a concise and convenient language by which to conduct the processes of reasoning necessary to the solution of mathematical questions. The illustrations of this elementary idea of the nature of Algebra lead to the formation of Equations, the rules for the reduction of which are developed in the third section. The process for putting a question into equation, the topic which next naturally occurs, is explained in the following section; and the illustrations given in relation to this lead the learner to see that in Algebra there are operations required, analogous to addition, subtraction, multiplication and division, in Arithmetic. The origin of these operations being thus seen, and the necessity for them felt, the learner is prepared for their full development, which is given in the two next sections.

These operations being well understood, the way is prepared for questions involving more complicated operations than those previously solved. Questions of this description are now introduced in the sixth section, and the rules for the reduction of equations involving two or more unknown quantities are fully explained.

With these resources, and the discipline derived from the solution of the previous particular questions, the learner is now introduced, in the eighth section, to the subject of Generalization, usually the most difficult topic to be encountered by a young mind. Here the object is, first, to lead the learner to see clearly the distinction between the process of reasoning he is led to pursue in the solution of a question, and the numerical operations he is required to perform as the result of that process. In order to this, he is required to perform numerous questions, retaining the operations as he proceeds, and leaving them all to be performed at last, when the reasoning process has reached its conclusion. In this way he is led to see that the reasoning process is precisely the same, and the operations to be performed precisely the same, for all questions which differ only in the particular numbers that are given, and that thus, in fact, he has obtained a general solution of his question. A little practice in this mode of generalization leads him to feel the want of general symbols to represent the given things in a question, and thus a clear and full idea of this difficult topic is at length obtained.

From every general solution, the learner derives a rule for the solution of all the particular questions involved in it. The nature of a *rule* being thus understood, he is led, in the next section, to investigate anew, by aid of the new instrument now in his hands, the most important rules of Arithmetic, such as Proportions, Fellowship, the Rule of Three, Position, &c. He is thus led to a more perfect understanding of these rules; and the numerous examples given under each serve as an extended review of the principles already acquired, in his previous study of Arithmetic. The difference, moreover, between Algebra and Arithmetic, and the peculiar province of each, is now fully understood.

Proceeding, next, in the tenth and eleventh sections, with what is more peculiarly the subject of Algebra, the mode of extracting the square root of algebraic quantities, the nature and solution of equations of the second degree, are there explained, with sufficient examples for illustration. The topic next introduced is the Indeterminate Analysis, one of the most elegant and important branches of Algebra, but which is very generally, as well as strangely, omitted in textbooks of Algebra. An elementary view of this subject is given in the twelfth section. In the next, the subject of logarithms is somewhat fully developed, and finally, in the last section, the rules of compound interest as an important application of them.

The author has given this brief view of his general plan, in the hope that teachers who may use his work will fully possess themselves of it, and enter into its spirit. His own experience has led to the clear conviction that this course is best adapted to give the learner just notions of the nature and powers of Algebra, and to excite an interest in the study of it. He has endeavored, in the prosecution of the work, to furnish all needed help to the beginner, and at the same time to avoid that excess of explanation that leaves but little for the exercise of the talents of the teacher, and less for those of the pupil.

It is presumed, in this treatise, that the learner is well acquainted with the fundamental operations of Arithmetic, and with the practical rules ordinarily contained in works on this subject. With this knowledge, and the mental discipline acquired in its attainment, he will meet, it is believed, in this treatise, with no difficulties which are not essentially inherent in the subject, and necessary to a healthy and vigorous exercise of his powers.

WILLIAM SMYTH.

BOWDOIN COLLEGE, *Nov.*, 1850.

SECOND EDITION.

In the present edition such alterations and additions have been made as, in the use of the work, have appeared adapted to its improvement. The favor with which the first edition has been received, as evinced by its rapid sale, and the numerous testimonials of eminent teachers by whom it has been adopted as a text-book, lead to the belief that the work is well adapted to its object, and that it will be found a valuable aid in disseminating more widely among the pupils in our common schools and academies a knowledge of the first principles of Algebra.

Sept., 1851. W. S.

CONTENTS.

ELEMENTARY ALGEBRA.

SECTION I.

1. THE learner has already become acquainted, in Arithmetic, with the use of the following signs:

1°. The sign $=$, which denotes *equality*, and is read *equal to;* thus, 7 times three $= 21$.

2°. The sign $+$, which denotes *addition*, and is read *plus;* thus, $7 + 3 = 10$.

3°. The sign $-$, which denotes *subtraction*, and is read *minus;* thus, $15 - 4 = 11$.

4°. The sign $\times$, which denotes *multiplication*, and is read *multiplied by;* thus, $7 \times 9 = 63$, or, more concisely, 7×9 is read, 7 *times* 9.

5°. The sign $\div$, which denotes *division*, and is read *divided by;* thus, $24 \div 6 = 4$.

2. Let it now be required to find the answer to the following question:

1. A gentleman once had 40 dollars; he spent a certain part of it, and found that he had three times as much left as he had spent. How much money had he spent?

In order to obtain the answer, we should reason upon this question thus:

1°. The money left is three times the money spent.

But the money spent added to the money left should be equal to \$40, the money he had at first; wherefore,

2°. The money spent added to three times the money spent, is equal to \$40; or,

3° Four times the money spent is equal to \$ 40; whence,

4°. Once the money spent, or the answer sought, is equal to \$40 divided by 4, or to \$10.

3. Let us now employ in this process the signs above explained, and the reasoning will then stand thus:

1°. The money left is 3 $\times$ the money spent.

2°. The money spent $+$ 3 $\times$ the money spent $=$ 40.

3°. 4 $\times$ the money spent $=$ 40.

4°. once the money spent $= 40 \div 4 = 10$.

The reasoning, it is evident, is rendered more concise by the use of these signs. But the phrase, "*the money spent*," which expresses the unknown quantity, or answer sought, being often repeated, the reasoning will be rendered still more concise, if we represent this also by a sign.

For this purpose, we may employ any convenient character or symbol. Let us take some letter of the alphabet, x, for example: thus, let x equal the money spent; then

1°. $3\,x$ will be the money left, and

2°. $x + 3 \times x = 40$.

3°. $4 \times x = 40$.

4°. $x = 40 \div 4 = 10$.

To express the multiplication of x by 3, we may use a period instead of the sign $\times$; thus, $3\,.\,x$. Or, we may simply write the 3 before the x; thus, $3\,x$. Division is also usually indicated by writing the number to be divided above a horizontal line, and the divisor beneath it, in the form of a fraction; thus,

$$\frac{40}{4} = 10.$$

2. Two gentlemen, A and B, are 63 miles apart, and travel the direct road toward each other until they meet; A travels 3 miles an hour, and B travels 4 miles an hour. In how many hours will they meet?

To obtain the answer, we should reason upon this question thus:

1°. The distance A travels will be equal to the number of hours in which they will meet, multiplied by 3.

2°. The distance B travels will be equal to the number of hours in which they will meet, multiplied by 4.

And since both must together travel over the whole distance they are apart, or 63 miles, then

3°. Three times the number of hours in which they will meet, added to four times the number of hours in which they will meet, must be equal to 63; or,

4°. Seven times the number of hours in which they will meet are equal to 63; whence,

5°. The number of hours in which they will meet is equal to 63 divided by 7, or 9, the answer sought.

Employing the signs above explained in this process, let us represent by x the phrase which denotes the unknown quantity or answer sought, viz., "the number of hours in which they will meet." The process will then stand thus:

1°. $3\,x =$ the distance A travelled.

2°. $4\,x =$ the distance B travelled.

3°. $3\,x + 4\,x = 63$

4°. $7\,x = 63$

5°. $x = 63 \div 7 = 9.$

We thus find that they will meet in 9 hours.

3. Said a gentleman to me, if with the money I now have, I had twice as much, and $14 more, I should have just $74. How much money had he?

Let x represent the unknown quantity, or answer sought, in this question also; then employing the other signs as above,

$$x + 2x + 14 = 74$$
$$3x + 14 = 74$$
$$3x = 74 - 14 = 60$$
$$x = \frac{60}{3} = 20.$$

ANS. He had $20.

If we now conduct in common language the reasoning process necessary to obtain the answer to this question, or if we translate into common language the process above, it will be seen by this, as by the preceding examples, how much more concisely the reasoning is expressed by the aid of the signs we have explained.

4. The great utility of these signs will be more obvious as we proceed to more complicated questions. The learner will now, however, be able to see an important distinction between Arithmetic and Algebra. In Arithmetic, we are taught the use of a convenient set of signs, called figures, in representing numbers and facilitating the operations to be performed upon them. In Algebra, we learn the use of another set of signs, in representing the phrases employed, and, in general, in facilitating the reasoning process required, in order to obtain the answer to a proposed question.

Thus Algebra aids us in the reasoning process upon a question. Arithmetic enables us to perform with ease the numerical operations which the reasoning process determines to be necessary in order to obtain the answer.

In the above examples we have used the letter x to represent the unknown quantity, or answer sought. With equal propriety we might have used any other letter, or convenient sign. It is common, however, to use some one of the last letters of the alphabet, as x, y, or z, &c., to represent the unknown quantity, or that which is required in a question.

The signs above explained, together with those which will be hereafter introduced, are called *Algebraic signs.*

SECTION II.

5. The following examples will now serve as an exercise in the use of the algebraic signs we have explained. Let the learner, in each case, compare the solution with that which would be required in the use of common language.

1. Two gentlemen arrived late in the evening at an inn. To obtain a room, one offered to pay double and the other three times the ordinary price. Both together paid $7. What was the ordinary price of a room?

Let $x =$ the ordinary price of the room; then double the ordinary price will be $2x$, three times the ordinary price will be $3x$, and we shall have

$$2x + 3x = 7$$
$$5x = 7$$
$$x = \frac{7}{5} = 1\frac{2}{5}.$$

The answer will be $ 1.40.

2. Three men, A, B and C, trade in company, and gain

\$408, of which B has twice and C three times as much as A. What is the share of each?

Let x represent A's share; then $2x = $ B's, and $3x = $ C's share; and, by the question, we have

$$x + 2x + 3x = 408$$
$$x = 68.$$

Ans. A's share is \$68, B's \$136, C's \$204.

3. Two gentlemen saw a sum of money lying on the table. The first said, "I have three times as much money as there is on that table." "Well," replied the other, "I have ten times as much." Both together had \$65. How much money was there on the table?

Let $x = $ the money on the table. Ans. \$5.

4. Two capitalists, A and B, calculate their fortunes. It appears that A is twice as rich as B, and that together they possess \$5400. What is the capital of each?

Let $x = $ B's capital. Ans. B's \$1800, A's \$3600.

5. In a company of 266 persons, consisting of officers, merchants and students, there were four times as many merchants and twice as many officers as students. How many were there of each class?

Ans. 38 students, 76 officers, and 152 merchants.

6. A merchant freighted a vessel of 360 tons with sugar, coffee and tea. He put on board twice as much coffee and three times as much sugar as tea. How many tons were occupied by each?

Ans. 60 by tea, 120 by coffee, and 180 by sugar.

7. A, B and C were talking of their ages. A said, "I am twice as old as B;" B said, "I am twice as old as C." Together, they were 140 years old. What are the ages of each?

Let $x =$ C's age; then $2x =$ B's and $4x =$ A's age

ANS. A's 80, B's 40, and C's 20 years.

8. Four persons, A, B, C and D, are to share $120 in such a manner that B is to receive double of A's share, C double of B's, and D double of C's share. What is the share of each?

ANS. A's share $8, B's $16, C's $32, and D's $64.

9. A complained that he owed as much again as he was worth; B avowed he owed 3 times as much as A, whereupon C remarked that his debt was 5 times as large as B's. All three owed, together, $76,000. How much was A worth, and how much did each owe?

ANS. A was worth $2000. A owed $4000, B $12,000, and C $60,000.

10. Three traders, A, B and C, make together a clear gain of $360, which is to be divided among them according to the stock each put in trade. Now, it is known that A has put in as much as B and C together, and that B has put in just twice as much as C. What is each one's share of the profits?

ANS. C's $60, B's $120, and A's $180.

11. A father dies, and leaves his property, amounting to $4800, to be divided among his four children, A, B, C and D, in the following manner: A, who is the youngest, is to receive as much as his three brothers, B, C and D, taken together; B, the next oldest, is to receive as much as C and D, taken together; but C and D are each to receive an equal share. What is the share of each.

ANS. D's $600, C's $600, B's $1200, A's $2400.

12. Four poor persons, A, B, C and D, are to share $552 according to their ages. It is known that D is as old

as A, B and C, together; C is as old as A and B, and B is twice as old as A. What is the share of each?

ANS. A's \$46, B's \$92, C's \$138, D's \$276.

13. Two brothers are to share an inheritance of \$1000 The oldest being able to provide for himself, the younger is to receive \$40 more than three times the share of the other. What is the share of each?

Let $x =$ the share of the oldest; then $3x + 40$ will be the share of the younger, and we have

$$x + 3x + 40 = 1000$$
$$x = 240.$$

ANS. The oldest receives \$240, and the younger \$760.

14. At an election two candidates presented themselves for office. The one that was elected received 40 votes more than the other; the number of votes cast amounted to 260. How many votes had each?

ANS. 110 and 150, respectively.

15. What two numbers are those whose sum is 59, and difference 17? ANS. 21 and 38.

16. Two gentlemen, A and B, had together a fortune of \$5400; but A has \$5000 less than B. How many dollars had each? ANS. A has \$200, B \$5200.

17. A person employed 4 workmen; to the first of whom he gave two shillings more than to the second, to the second 3 shillings more than to the third, and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Let $x =$ the sum received by the fourth; then $x + 4$, $x + 7$, $x + 9$, will be the sums received by the third, second and first, respectively; and we have

$$x + x + 4 + x + 7 + x + 9 = 32$$

Here x, or the quantity sought, is repeated four times; and we have, therefore,

$$4x + 4 + 7 + 9 = 32.$$

And since the numbers 4, 7 and 9, are all additive, instead of adding them separately, we may add their sum, which gives

$$4x + 20 = 32,$$

whence

$$x = 3.$$

Ans. They received 12, 10, 7, and 3 shillings, respectively.

18. A certain estate, amounting to \$10,000, is to be divided among three children in the following manner: The second is to receive as much as the eldest and \$500 more, and the youngest is to receive three times as much as the eldest and \$300 more. What will each receive?

Ans. \$1840, \$2340, and \$5820, respectively.

19. A company of 90 persons consists of men, women and children. The men are 4 in number more than the women, the children 10 more than the adults. How many men, women and children, are there in the company?

Ans. 22 men, 18 women, and 50 children.

20. A poor man had 6 children; the eldest of which could earn 7*d*. a week more than the second, the second 8*d*. more than the third, the third 6*d*. more than the fourth, the fourth 4*d*. more than the fifth, and the fifth 5*d*. more than the youngest. They all together earned 10*s*. 10*d*. a week. How much could each earn a week?

Ans. 38, 31, 23, 17, 13, and 8 pence a week.

21. It is required to divide the number 99 into 5 such parts, that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

Let $x =$ the first part;

then $x - 3 =$ the second,

$x + 10 =$ the third,

$x - 9 =$ the fourth,

$x + 16 =$ the fifth,

and we have

$$x + x - 3 + x + 10 + x - 9 + x + 16 = 99.$$

Here x is repeated 5 times, the sum of the numbers to be added is 26, and the sum of those to be subtracted is 12; we shall have, therefore,

$$5x + 26 - 12 = 99,$$

or $$5x + 14 = 99;$$

whence $$x = 17.$$

ANS. The parts are 17, 14, 27, 8, and 33.

22. \$100 are to be divided among three persons, A, B and C, in such a manner that A is to receive \$20 more and B \$13 less than C. What is the share of each?

ANS. A's \$51, B's \$18, and C's \$31.

23. A clerk was 6 years in the same house. In the first 3 years he spent only \$300 a year, but in each following year \$100 more than in the preceding year; and yet, at the end of the sixth year, he had saved \$2400. What was his salary? ANS. \$800.

24. What number is that, from the treble of which if 18 be subtracted, the remainder is 6? ANS. 8.

25. A farmer sold 13 bushels of barley at a certain price, and afterwards 17 bushels at the same price, and at the second time received 36 shillings more than at the first. What was the price of a bushel? ANS. 9 shillings.

26. A person bought 198 gallons of beer, which exactly filled four casks; the first held twice as much as the second, the second twice as much as the third, and the third three

times as much as the fourth. How many gallons did each hold? ANS. 108, 54, 27, and 9 gallons.

27. What two numbers are there whose sum is 79, and difference 27? ANS. 26 and 53.

28. A draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each?

ANS. 35, 50 and 74 yards, respectively.

29. A gentleman being asked the age of his son, answered, My wife is 23 years older than my son, I am 10 years older than my wife, and all three are 110 years old. What is the age of the son? ANS. 18.

30. A father, who has three sons, leaves them 15,000 crowns. The will specifies that the second shall have 2000 crowns less than the eldest, and the youngest 1000 crowns more than the second. What is the share of each?

ANS. 6000, 4000, and 5000 crowns, respectively.

31. Four men, A, B, C and D, found a purse of money, containing $260; but not agreeing about the division of it, each took as much as he could get. A got a certain sum; B got 5 times as much, wanting $10; C, 7 times as much and $15 more; and D, as much as B and C both. How many dollars did each get?

ANS. A $10, B $40, C $85, and D $125.

32. There are five towns, in the order of the letters A, B, C, D, E. From A to E is 80 miles; the distance between B and C is ten miles more, between C and D is 15 miles less, and between D and E 17 miles more, than the distance between A and B. What are the respective distances? ANS. 17, 27, 2, and 34 miles.

33. To find a number such that if one-third of this number be added to itself the sum will be 60.

Let $x =$ the number. Then $x \div 3$, or $\frac{x}{3} =$ one-third of x; and, by the question,

$$x + \frac{x}{3} = 60;$$

or, reducing to the same denominator,

$$\frac{4x}{3} = 60;$$

whence $4x = 180$, and $x = 45$.

From what has been done, we see how quantities may be represented by algebraic signs. The following exercises will serve as a review of this part of the subject:

If a number be represented by x, what will represent a number greater by 4? what a number less by 7? what a number 5 times as large? what a number 10 times as large and 12 more?

If two numbers are expressed by $5x$, $3x$, respectively, what single term will express their sum? what their difference?

If three quantities are expressed by $2x$, $5x$, $7x$, respectively, what single term will express their sum? what the difference between the sum of the two first and the last?

If a number be represented by x, what will represent one-seventh of this number? what three-sevenths? what two-ninths? what five-thirteenths? what once and a third? what twice and three-fifths? what four times and seven-eighths?

SECTION III.—EQUATIONS.

6. IF the learner examines with attention the examples thus far performed, he will see that the first thing we have done has been, by aid of the algebraic signs, to obtain an expression for the equality of two things. Thus, in the third example, Art. 1, the expression first obtained was

$$x + 2x + 14 = 74.$$

This expression of the equality of two things which are equal is called an *equation*. The parts on either side of the sign of equality are called the *members* of the equation.

Thus, in the equation above, the part $x + 2x + 14$, on the left of the sign of equality, is called the left-hand member of the equation, and the part 74, on the right, is called the right-hand member.

The parts separated by the signs $+$ or $-$ are called *terms*. Thus, in the equation above, x, $2x$, &c., are terms of the equation.

A figure written before a letter, showing how many times the letter is taken, is called the *coefficient* of that letter. Thus, in the quantities $2x$, $5x$, 2 and 5 are the coefficients of x.

An equation is said to be *verified* when the value found for the unknown quantity, substituted in the equation, renders the two members identically the same.

Thus, in the equation $x + 2x + 14 = 74$, the value for x is found to be 20. Substituting this for x in the equation, it becomes

$$20 + 40 + 14 = 74,$$

or

$$74 = 74;$$

20 is, therefore, the true answer.

7. The process by which this first equation is obtained is called *putting the question into an equation.* No precise rule can be given by which, in all cases, a question can be put into an equation. In general, it must be done by a careful consideration of the conditions of the question, and the skilful use of the signs by which our reasoning upon them is facilitated.

When, however, the equation of a question has been obtained, there are regular steps by which the value of the unknown quantity is obtained from it, which we will now explain.

In order to this, let there be the following equation:

$$x = 5. \quad (1)$$

If to the left-hand member of this equation we add an x, and its value, 5, be added to the right-hand member, the equality will be preserved; and, we shall have

$$2\,x = 10. \quad (2)$$

If now we subtract an x from the left-hand member of this last, and its value, 5, from the right-hand member, the equality will be preserved; and, in general,

1°. *If the same or equal quantities be added to both members of an equation, the equality still continues.*

2°. *If the same or equal quantities be subtracted from both members of an equation, the equality still continues.*

From which it follows:

3°. *If both members of an equation are multiplied by the same or equal quantities*, or

4°. *If both members of an equation are divided by the same or equal quantities, the equality still continues.*

These principles require no demonstration. They should, however, be clearly understood, since they serve as the basis of all operations upon equations.

8. This being premised, let now the following equation be proposed, viz.:

$$x + \frac{x}{3} + \frac{x}{5} = 92. \quad (1)$$

The process by which the value of x, or the unknown quantity, is obtained from an equation, is called *resolving* or *reducing* the equation.

To resolve the proposed equation, we first free the fractional terms from their denominators.

In order to this, we multiply both sides of the equation successively by the denominators, which will not destroy the equality. Beginning with 3, the denominator of the second term, we obtain

$$3x + x + \frac{3x}{5} = 276; \quad (2)$$

multiplying next by 5, we obtain

$$15x + 5x + 3x = 1380, \quad (3)$$

an equation free from denominators.

Uniting terms in this last, $23x = 1380$; (4)

whence, dividing both members by 23,

$$x = 60. \quad (5)$$

If now, in order to verify the result, we substitute 60 for x in the proposed, we obtain

$$60 + \frac{60}{3} + \frac{60}{5} = 92,$$

or

$$92 = 92;$$

which shows that 60 is the true answer.

2. What is the value of x in the equation

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{5} - \frac{x}{7} = 187?$$

Ans. $x = 210$.

3. What is the value of x in the equation

$$\frac{x}{2}+\frac{x}{4}-\frac{x}{3}+\frac{x}{9}=20?$$

In this example it will be seen that 4 is a multiple of 2 that is, it is divisible by 2. If, then, we begin by multiplying by 4, the numerator of the first term will be divisible by its denominator, and, the operation being performed, will be left free of its denominator, and the equation will stand

$$2x+x-\frac{4x}{3}+\frac{4x}{9}=80.$$

Again, since 9 is divisible by 3, multiplying next by 9, we obtain $18x+9x-12x+4x=720$;
whence $x=37\frac{17}{19}$.

By multiplying first by 4 and then by 9, the equation is freed from its denominators by two multiplications only, instead of four; and is moreover left, when freed from denominators, in a more simple state.

4. What is the value of x in the equation

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{6}-\frac{x}{7}+\frac{x}{21}=38?$$

ANS. $x=42$.

5. What is the value of x in the equation

$$\frac{x}{2}+\frac{2x}{5}-\frac{3x}{4}+\frac{7x}{10}=25\frac{1}{2}?$$

ANS. $x=30$.

6. What is the value of x in the equation

$$\frac{3x}{4}-\frac{5x}{6}+\frac{x}{12}+\frac{2x}{3}=64?$$

ANS. $x=96$.

7. What is the value of x in the equation

$$\frac{x}{2}+\frac{2x}{3}-\frac{3x}{14}+\frac{5x}{12}-\frac{6x}{7}+\frac{x}{21}=70\frac{1}{2}?$$

The least number divisible by each one of the denominators, or, in other words, their least common multiple, is 84. Multiplying by 84, and dividing each fractional term, as we proceed, by its denominator, we obtain

$$42x + 56x - 18x + 35x - 72x + 4x = 5922.$$

The equation is thus freed of its denominators by one multiplication, and is moreover left in its most simple state.

Deducing next the value of x, we obtain

$$x = 126.$$

By the use of the least common multiple, an equation, it is evident, may be freed at once of its denominators. As this process, however, in most cases involves a multiplication by a large number, it is in general most convenient in practice to multiply by the denominators successively, taking care to commence the operation with such denominators as are divisible by some one or more of the others, or which contain factors in common with them.

Thus, in the last example, multiplying first by 12 and reducing, we have

$$6x + 8x - \frac{18x}{7} + 5x - \frac{72x}{7} + \frac{4x}{7} = 846.$$

Multiplying next by 7, the equation is freed of its denominators, as before.

8. What is the value of x in the equation

$$\frac{x}{6} - \frac{x}{4} - \frac{x}{3} + \frac{x}{2} = 1?$$

ANS. $x = 12$.

9. What is the value of x in the equation

$$\frac{x}{2} + \frac{2x}{3} - \frac{x}{5} + \frac{3x}{10} + \frac{x}{12} = 81?$$

ANS. $x = 60$

10. What is the value of x in the equation

$$\frac{x}{3} - \frac{3x}{5} + \frac{4x}{7} + \frac{x}{10} + \frac{3x}{14} = 5\tfrac{5}{12}?$$

Ans. $x = 8\frac{3}{4}$.

9. Let it next be required to find the value of x in the following equation:

$$8x - 72 = 3x. \quad (1)$$

To resolve this equation, we change the places of the terms, so that the unknown quantity will stand on one side of the equation and the known quantities on the other.

In order to this, we add 72 to both sides, which gives

$$8x - 72 + 72 = 3x + 72,$$

or, reducing,

$$8x = 3x + 72.$$

Subtracting next $3x$ from both sides, we obtain

$$8x - 3x = 3x - 3x + 72,$$

or,

$$8x - 3x = 72. \quad (2)$$

If now we compare equation (2) with equation (1), it will be seen that the 72, which has the sign $-$ before it in the left-hand member, has been made to disappear from that member, and to appear in the right-hand member with the sign $+$ before it. In like manner the term $3x$, which has the sign $+$ understood before it in the right-hand member, has been made to disappear from that member, and to appear in the opposite with the sign $-$ before it. The result is the same as if we had transferred each of these terms from the members in which they stood at first to the opposite, and changed their signs.

The process by which a term is transferred from one side of an equation to the other is called *transposition*, and we have the following rule, viz.: *Any term may be transposed from one side of an equation to the other, if at the same time we change its sign.*

2. What is the value of x in the equation

$$4x + 22 = 57 - 3x?$$

Transposing, $4x + 3x = 57 - 22$;

whence $x = 5$.

3. What is the value of x in the equation

$$336 + 3x - 11 = 10x - 10 + 776 - 56x?$$

ANS. $x = 9$.

4. What is the value of x in the equation

$$36x - 12 + 3x - 48 = 20x + 150 - 11x?$$

ANS. $x = 7$.

5. What is the value of x in the equation

$$51 - 9x - 10 - 20x = 75 - 90x + 70 + 35x?$$

ANS. $x = 4$.

10. Let it be proposed next to find the value of x in the following equation:

$$\frac{x}{2} + \frac{2x}{3} - \frac{5}{6} = \frac{4x}{5} + 45.$$

Freeing from denominators,

$$15x + 20x - 25 = 24x + 1350;$$

transposing, $15x + 20x - 24x = 1350 + 25$;

reducing, $11x = 1375$;

dividing by the coefficient, $x = 125$.

Equations are distinguished by different degrees. Equations such as the preceding, in which the unknown quantity is neither multiplied by itself nor by any other unknown quantity, are called equations of *the first degree*.

The rules obtained above are sufficient for the solution of all equations of this kind. When there is but one unknown quantity, the process is as follows:

1°. *Free the equation from denominators.* 2°. *Collect on one side the terms which contain the unknown quantity, and* *those which are known on the other.* 3°. *Reduce to one term the terms in each member.* 4°. *Divide by the coefficient of x.*

11. The learner will now be prepared to solve the following questions, which will serve as an additional exercise in the reduction of equations.

1. A draper had a piece of cloth, from which he sold a third and a fourth part, which he found to be equal to 25 yards. How many yards were there in the whole piece?

ANS. $42\frac{6}{7}$.

2. A gentleman, being asked how much money he had in his pocket, answered, the fourth and fifth part amounted together to 9 dollars. How much money had he?

ANS. $20.

3. Two gentlemen wished to buy a horse. Upon counting their money, they found that one had but one-seventh and the other one-ninth of the whole money that was asked for him; and yet they had together $32. What was the price of the horse? ANS. $126.

4. Three shepherds, A, B and C, upon counting their sheep, find that B has three times as many as A, but that C has only one-fourth as many as B. They have in all 19 sheep. How many had each?

ANS. A had 4, B had 12, and C had 3.

5. A gentleman put out his money at three different places. At the first place he puts one-seventh, at the second one-ninth, and at the third one-half. It is known that at the last place he has $155 more than at the other two taken together. What is the whole amount of his money?

ANS. $630.

6. A young man, being asked his age, answered, if one-half, two-thirds, three-fourths, four-fifths and five-sixths of

his age, and 9 years more, were added to his age, he would be 100 years old. How old was he? ANS. 20 years.

7. To find a number such that if 7 be subtracted from five times the number, and 8 be added to twice the number, the remainder will be equal to the sum.

ANS. The number is 5.

8. Three men, A, B and C, make a joint contribution. A puts in a certain sum, B puts in twice as much as A and \$12 more, C puts in \$10 less than five times as much as A; when it was found that C had contributed as much as A and B together. How much did each contribute?

ANS. A \$11, B \$34, and C \$45.

9. An officer, giving an account of an engagement with the enemy, reported that half the men he commanded were taken prisoners, one-fourth were killed, one-seventh severely wounded, and that in consequence of this he had but 3 men left for service. How many men had he before the engagement? ANS. 28.

10. A gentleman going to market put a certain sum of money in his pocket: one-third of it he paid for sugar, one-fourth for coffee, one-sixth for tea, one-seventh for rice, and the remainder, amounting to \$$9\frac{1}{5}$, for other articles. How much money did he put in his pocket? ANS. \$$85.86\frac{2}{3}$.

11. A post is one-fourth of its length in the mud, one-third in the water, and 10 feet above the water. What is its length? ANS. 24 feet.

12. A shepherd drove himself one-half of his flock to pasture; one-fourth of them were driven by his son, one-eighth by his daughter, and 40 remained in the stable. How many had he? ANS. 320.

13. In a mixture of copper, tin, and lead, 16 lbs. less than one-half the whole was copper, 12 lbs. less than one-third

the whole was tin, and 4 lbs. more than one-fourth the whole was lead. What quantity was there of each in the mixture?

ANS. 128 lbs., 84 lbs., and 76 lbs., respectively.

SECTION IV.—PUTTING A QUESTION INTO AN EQUATION.

12. We have seen the manner in which an equation may be reduced, after it has once been formed. It will be seen also that, in obtaining the equations in the preceding examples, *we have represented the answer sought by a letter, and have then indicated upon it the operations necessary to prove it to be the true answer, supposing it to be known.*

No certain rule can be given for putting a problem into an equation. The process pursued above will, however, generally lead to the equation. We will illustrate this by some additional examples.

1. After a certain part of a commodity, which weighed 40 lbs., was sold, there still remained 8 lbs. more than the quantity sold. How many pounds were there sold?

In order to prove the answer, if it were given, we should first subtract it from 40: we should next add 8 to it, and if the difference and sum were found to be equal, the answer would be correct.

Imitating this process, let us put x for the answer. Subtracting x from 40 gives $40 - x$; adding 8 to x gives $x + 8$; putting next these two results equal to each other, we have

$$40 - x = x + 8.$$

This is the equation of the question, which being resolved gives $x = 16$ for the answer.

2. A gentleman remarked, If I had three times as much money as I have, I should have $18 more than I now have. How much money had he? ANS. $9.

3. A bookseller sold 10 books at a certain price, and afterwards 15 more at the same rate. Now, at the latter time he received 35 shillings more than at the first. What did he receive for each book? ANS. 7*s.*

4. A father, wishing to divide the money he had in his purse between his sons, found that he wanted 5 shillings to be able to give them 4 shillings each. He therefore gave them 3 shillings only, and found that he had 7 shillings left. How many sons had he?

To prove the answer, if it were known, we should multiply it first by 4, and subtract 5 from the product; we should next multiply it by 3, and add 7 to the product. If the two results were equal, the answer would be right.

To imitate this process, let $x =$ the answer. Multiplying x by 4 gives $4x$, and subtracting 5 from this gives $4x - 5$. Multiplying next x by 3 gives $3x$, and adding 7 to this gives $3x + 7$. Putting these results equal, we have, for the equation of the question,

$$4x - 5 = 3x + 7;$$

from which we obtain $x = 12.$

5. A person wishing to buy some sugar found that if he gave but 7*d.* per lb., he would have 10*d.* left; but if he gave 8*d.* per lb., he would lack 2*s.* 6*d.* How many pounds of sugar did he buy? ANS. 40 lbs.

6. A person wishing to buy a house, draws $250 from each of his debtors, and finds that in this case he has not enough to make the purchase by $2000; but if he draws $340 from each, he has $880 more than he needs. How many debtors has he? ANS. 32.

7. A person having received \$20, went into a store to buy himself some broadcloth for a cloak. The store-keeper showed him two different kinds. If he bought of the first, which was \$3 a yard, he would have as many dollars left as he would lack if he had wished to buy of the second piece, which was \$5 a yard. How many yards did he wish to buy? ANS. 5 yards.

8. In a certain school, one-third of the pupils study Arithmetic, one-fifth Geography, and the remainder, consisting of 49, study Algebra. How many pupils are there in the school? ANS. 105.

9. The head of a fish is eleven inches long; its tail is as long as its head and half its body, and its body is as long as its head and tail. What is its length?

ANS. 7 feet 4 inches?

10. One-fourth of the contents of a cask leaked out; ten gallons and a half were afterwards drawn from it, when the cask was found to be two-thirds full. What was the whole content of the cask? ANS. 126 gallons.

11. A courier, who travels 60 miles a day, had been despatched 5 days, when a second was sent to overtake him; in order to which, he must travel 75 miles a day. In how many days will he overtake the former?

In order to verify the answer, if it were known, we should add 5 to it, and multiply the sum by 60; we should then multiply it by 75, and if these two products were found to be equal, the answer would be right.

Imitating this process, let $x =$ the answer; adding 5 to it, we have $x + 5$; this multiplied by 60 should be equal to $75x$.

The multiplication of $x + 5$ by 60 is indicated by enclosing the $x + 5$ in a parenthesis, and writing the 60 outside; thus, $60(x + 5)$; whence we have for the equation,

$$60\,(x + 5) = 75\,x.$$

To perform the multiplication indicated, both terms, it is evident, should be multiplied by 60; that is, the product is equal to 60 times 5 added to 60 times x; whence,

$$60\,x + 300 = 75\,x;$$

from which we obtain $x = 20.$

12. Two men talking of their ages, the first says, Your age is 18 years more than mine, and three times your age is equal to five times mine. What is the age of each?

ANS. 27 and 45 years.

13. A man bought 7 cows and 11 oxen for \$327. For the oxen he gave \$15 apiece more than for the cows. How much did he give apiece for each?

ANS. For the cows, \$9; for the oxen, \$24.

14. A man, when he was married, was three times as old as his wife. After they had lived together 20 years, he was only twice as old. What was the age of each?

ANS. 20, and 60 years.

15. A farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93, and finds just twice as many remaining in one as in the other. How many did each flock originally contain?

If the answer to this question were known, we should prove it by subtracting first 39 from it, and then 93; we should then double this last remainder, and if it were equal to the first, the answer would be right.

To imitate this process, let $x =$ the answer; subtracting first 39 from x, we have $x - 39$; subtracting next 93 from it, we have $x - 93$; this last remainder, multiplied by 2, should be equal to the first remainder, or $x - 39$.

The multiplication of $x - 93$ by 2 is indicated by enclosing the $x - 93$ in a parenthesis, and writing the 2 outside of it, thus: $2(x - 93)$; we have then, for the equation of the question, $x - 39 = 2(x - 93)$.

But how shall we multiply $x - 93$ by 2? Since x is diminished by 93, if we multiply the x only by 2, the result will be too great by 93 multiplied by 2. To express the true product, therefore, from $x \times 2$, or $2x$, we subtract 93×2, or 186, and the equation becomes

$$x - 39 = 2x - 186;$$

from which we obtain $x = 147$.

If $149 - 7x$ be multiplied by 3, how shall we indicate the product, and what will be its value?

ANS. $3(149 - 7x) = 447 - 21x$.

Indicate the product, and find its value, in the following cases:

1. $48 + x$ multiplied by 9.
2. $105 + 2x$ multiplied by 37.
3. $x + 53$ multiplied by 4.
4. $59 - x$ multiplied by 5.
5. $123 - 4x$ multiplied by 7.
6. $254 - 7x$ multiplied by 17.
7. $x - 73$ multiplied by 13.
8. $5x - 125$ multiplied by 9.
9. $x - 39$ multiplied by 8.
10. $x - 25$ multiplied by 3.

16. Divide the number 197 into two such parts that four times the greater may exceed five times the less by 50.

ANS. The parts are 82 and 115.

17. A certain sum is to be raised upon two estates, one of which pays 19 shillings less than the other; and if 5 shillings be added to treble the less payment, it will be equal to twice the greater. What are the sums paid?

ANS. 33 and 52 shillings.

18. Bought 12 yards of cloth for £10 14*s*. For part of it I gave 19 shillings a yard, and for the rest 17 shillings a yard. How many yards of each did I buy?

ANS. 5 yards at 19*s*., and 7 at 17*s*.

19. A mercer, having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants taken together were 142 yards. What was the length of each piece? ANS. 54 yards.

20. A gentleman employed two laborers at different times, one for 3 shillings and the other for 5 shillings a day. Now, the number of days added together was forty, and they each received the same sum. How many days was each employed?

ANS. The first 25, and the second 15 days.

21. There are two numbers, one of which is four-fifths of the other; but, by subtracting 75 from each, the remainder of the larger is twice as great as the remainder of the smaller. What are the numbers? ANS. 125 and 100.

22. The sum of two numbers is 36; but when the greater is multiplied by 4 and the smaller by 3, the difference between the two products is 32. What are the two numbers?

If one of the numbers, the greater, for example, were known, in order to verify it, we should subtract it first from 36; this would give the less number. Then subtracting three times the less from four times the greater, the remainder should be equal to 32.

To imitate this process, let $x =$ the greater; then $36 - x$ will be the less; four times the greater is $4x$, and three times the less is $3(36 - x)$; the latter product subtracted from the former should be equal to 32. Indicating this subtraction, we have, for the equation of the question

$$4x - 3(36 - x) = 32;$$

or, performing the multiplication,

$$4x - (108 - 3x) = 32.$$

But how shall we perform the subtraction required in this equation? It is evident that, since 108 ought to be diminished by $3x$ before subtraction, if we begin by subtracting 108, we shall take away too much by $3x$; we must add $3x$, therefore, to the remainder thus obtained, in order to have the true remainder.

Performing these operations, we obtain

$$4x - 108 + 3x = 32;$$

whence $$x = 20.$$

How shall we express the value of $100 - x$ subtracted from $3x$, and what will be the value of the expression when reduced?

Ans. $3x - (100 - x)$, which is equal to $4x - 100$.

How shall we indicate the subtraction, and what will be the value of the reduced expressions, in the following cases?

1. $5x$ diminished by $75 - 3x$.
2. $9x$ diminished by $83 - 7x$.
3. 100 diminished by $3x - 25$.
4. 230 diminished by $75 - 4x$.
5. 84 diminished by $35 - 10x$.
6. $3x - 7$ diminished by $7x - 50$.
7. $45x + 20$ diminished by $29 - 22x$.

23. Divide the number 20 into two such parts that twice the greater may exceed three times the less by 5.

Ans. The parts are 13 and 7.

24. The sum of two numbers is 40; and if the greater be multiplied by 3 and the less by 5, the difference of the products will be 24. What are the numbers?

Ans. 28 and 12.

25. The difference of two numbers is 25; and if twice

the less be taken from three times the greater, the remainder will be 80. What are the numbers? ANS. 30 and 5.

26. The sum of 75 dollars is to be divided between two poor persons, so that three times what one receives exceeds seven times what the other receives by $15. What did each receive? ANS. 54 and 21 dollars.

27. A man has a horse and chaise, which together are worth $400. Now, if the value of the chaise be subtracted from twice that of the horse, the remainder will be the same as if three times the value of the horse were subtracted from twice that of the chaise. Required the value of each.

ANS. $150, and $250, respectively.

28. A person at play won twice as much as he began with, and then lost 19 shillings. After this he lost one-fifth of what remained, and then won as much as he began with, and, counting his money, found he had 80 shillings. What sum did he begin with?

Let $x =$ the number of shillings he began with; then $3x =$ the sum he had after winning $2x$, and $3x - 19 =$ the sum remaining after the first loss. Now, since he lost next one-fifth of this, he will have four-fifths of it remaining. One-fifth of $3x - 19$ is expressed thus:

$$\frac{3x - 19}{5},$$

a line being drawn under the $3x - 19$, and the divisor placed beneath it, in the form of a fraction. Four-fifths of it will, therefore, be expressed thus: $\frac{4(3x - 19)}{5}$,

and we have for the equation of the question,

$$\frac{4(3x - 19)}{5} + x = 80;$$

whence, $x = 28$.

29. A and B have together a fortune of \$9800. A puts out one-sixth of his money, B only one-fifth of his, when it is found that each has the same sum remaining on hand. What is each man's fortune?

ANS. A's \$4800 and B's \$5000.

30. Says A to B, I have \$12 more than you, and two-thirds of my money is equal to three-fourths of yours. How much money had each? ANS. \$108 and \$96.

31. A man bought a horse and saddle; for the horse he gave \$180 more than for the saddle, and four times the price of the saddle was equal to two-fifths the price of the horse. What was the price of each?

ANS. Saddle \$20, horse \$200.

32. Two men, A and B, set out on a journey, each with the same sum of money. A spends \$40 and B \$30; then three-eighths of A's money, subtracted from five-sevenths of B's, equals one-fourth of what each carried from home. How much money had each, on commencing the journey?

By the question,

$$\frac{5(x-30)}{7}-\frac{3(x-40)}{8}=\frac{x}{4}.$$

Freeing from denominators,

$$40(x-30)-21(x-40)=14x;$$

whence, $40x-1200-21x+840=14x;$

from which we obtain $x=72.$

2. What is the value of x in the equation

$$\frac{4(x-10)}{5}-\frac{2(x-15)}{3}=20?$$

ANS. $x=135.$

3. What is the value of x in the equation

$$\frac{3(x+15)}{7}-\frac{4(x-20)}{9}=\frac{3x}{4}?$$

ANS. $x=20.$

4. What is the value of x in the equation

$$\frac{5(3x-5)}{3}-\frac{3(x+7)}{4}=4x?$$

Ans. $x=54\frac{1}{3}$.

33. A and B commence trade with equal sums of money; A loses \$90 and B loses \$70, when it is found that three-fourths of what A has left exceeds two-thirds of what B has left by \$50. With what sum did they commence trade?

Ans. \$850.

34. A vintner has two equal casks, full of wine; he draws 20 gallons out of one and 30 out of the other, when he finds that three-fifths of what remains in the first exceeds five-elevenths of what remains in the second by 30 gallons. How many gallons does each cask hold?

Ans. 195 galls.

35. Two merchants commence trade, each with the same capital; A loses \$1500, B gains \$250, when it is found that one-half of what A has left exceeds one-fifth of what B now has by \$160. What is the capital with which they commenced trade? Ans. \$3200.

36. Divide the number 30 into two such parts that the less may be to the greater as 2 to 3.

Two numbers are said to be in the proportion of 2 to 3, or simply as 2 to 3, when the first is two-thirds of the second, or the second is three-halves of the first; or, which is the same thing, when twice the second is equal to three times the first.

Let $x=$ the less part; then the greater will be $\frac{3x}{2}$,

and we shall have $x+\frac{3x}{2}=30$.

Ans. The parts are 12 and 18.

37. Two men found a purse containing $80, which they agreed to divide among themselves in the proportion of 3 to 5. What was the share of each? ANS. $30 and $50.

38. A prize of 2000 guineas was divided between two persons, A and B; their shares were in the proportion of 7 to 9. What were their shares?

ANS. A's 875, B's 1125 guineas.

39. Divide the number 44 into two such parts that the greater, increased by 5, may be to the less, increased by 7, as 4 is to 3.

Let $x =$ the greater; then $44 - x =$ the less; and we shall have $3(x + 5) = 4(51 - x)$.

ANS. The parts are 17 and 27.

40. A bankrupt owed to two creditors $140; the difference of the debts was to the greater as 4 to 9. What were the debts? ANS. $90 and $50.

41. A father's age is to that of his son as 5 to 2, and the difference of their ages is 30 years. What are their ages?

ANS. 50 and 20 years.

42. A man's age, when he was married, was to that of his wife as 4 to 3; but after they had been married 10 years, his age was to hers as 5 to 4. How old was each at the time of their marriage?

ANS. 40 and 30 years, respectively.

43. It is required to divide the number 34 into two such parts that the difference between the greater and 18 shall be to the difference between 18 and the less as 2 to 3.

ANS. The parts are 22 and 12.

44. Two men have now equal sums of money; but if one gives to the other $40, the sum the former then has will be to that which the latter has as 4 to 9. What sum did they have at first? ANS. $104.

45. A merchant in trade gained the first year $550, but the second year lost one-third of what he then had; after which he found that his stock was to that with which he began as 7 to 5. What was the stock with which he began?

ANS. $500.

46. From two casks of equal size are drawn quantities which are in the proportion of 6 to 7; and it appears that if 16 gallons less had been drawn from that which is now the emptier, only half as much would have been drawn from it as from the other. How many gallons were drawn from each?

ANS. 24 and 28.

47. Three men engaged in trade, and put in stock in proportion to the numbers 2, 3 and 5; that is, as often as A put in $2, B put in $3 and C $5. They gained $750. What was each man's share of the gain?

ANS. A's $150, B's $225, C's $375.

48. A cistern is supplied by two pipes: the first will fill it in three hours, the second in four hours; in how many hours will it be filled, if both run together?

In order to verify the answer to this question, we should calculate what part of the cistern would be filled by each of the pipes in the given time; the sum of these parts would be equal to the whole cistern, if the answer were correct.

To imitate this process, let $x =$ the time in which the cistern would be filled, if both pipes ran together. The capacity of the cistern being represented by unity, since the first pipe will fill it in three hours, in one hour it will fill one-third of it, and in x hours it will fill x-thirds of it. In like manner, in x hours, the second pipe will fill x-fourths of it. Then, as these two parts of the cistern should be equal to the whole of it, we shall have for the equation of the

question $$\frac{x}{3}+\frac{x}{4}=1;$$
whence $$x=1\tfrac{5}{7}.$$

49. Two masons are employed to build a wall in which there are to be 82 cubic feet. The first can build 7 cubic feet in 5 days, and the second 4 cubic feet in 3 days. In how many days can both together build the wall?

ANS. 30.

50. Three men, A, B and C, are to build a certain fence. A can build it in 3 days, B can build it in 4, and C in 5 days. How many days will they need to build the fence together? ANS. $1\frac{13}{47}$ days.

51. A man and his wife did usually drink out a vessel of beer in 12 days; but when the man was from home, the vessel lasted the woman 30 days. In how many days would the man alone drink it out?

Let x = the number of days; then the man would drink one-xth part in one day, and the woman one-thirtieth. The parts which each would drink in 12 days should be equal to the whole vessel. ANS. 20 days.

13. The examples which follow will serve as an additional exercise upon the principles thus far explained. They are arranged, for obvious reasons, without regard to the order in which these principles have been developed.

1. A father, taking his four sons to school, divided a certain sum among them. Now, the third had 9 shillings more than the youngest, the second 12 shillings more than the third, and the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

ANS. 21, 30, 42 and 60 shillings, respectively.

2. A sum of money was divided between two persons, A

and B, so that the share of A was to that of B as 5 to 3 while it exceeded five-ninths of the whole by 50 pounds What was the share of each person?

ANS. 450 and 270 pounds.

3. A gentleman started on a journey with a certain sum of money; after having had $60 stolen from him, he expended one-third of what he had left, and found that the remaining two-thirds wanted $90 to be equal to the sum he carried from home. How much money had he on commencing his journey? ANS. $150.

4. A person has a lease for 99 years; and being asked how much of it had expired, he replied that two-thirds of the time past was equal to four-fifths of the time to come. How many years had the lease to run? ANS. 45.

5. A gentleman has two horses and a chaise worth $150. Now, if the first horse be harnessed, the horse and chaise together will be worth twice as much as the second horse; but if the second horse be harnessed, the horse and chaise together will be worth three times as much as the first horse. What is the value of each horse?

ANS. $90 and $120.

6. A laborer agreed to work for a gentleman a year for $72 and a suit of clothes; but at the end of 7 months he was dismissed, having received his clothes and $32. What was the value of the clothes? ANS. $24.

7. A cistern has three cocks; the first will fill it in 5 hours, the second in 10 hours, and the third will empty it in 8 hours. In what time will the cistern be filled, if all the cocks are running together? ANS. $5\frac{5}{7}$ hours.

8. A man wished to enclose a piece of ground with palisadoes. He found that if he set them a foot asunder, he

should have too few by 150; but if he set them a yard asunder, he should have too many by 70. How many had he? Ans. 180.

9. Some persons agreed to give sixpence each to a waterman for carrying them from London to Gravesend; but with this condition, that for every other person taken in by the way threepence should be abated in their joint fare. Now, the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but fivepence each. How many persons were there at first? Ans. 36.

10. A sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and eight hours afterwards, B sets out from the same place, and travels the same road at the rate of 5 miles in 3 hours. How long and how far must A travel before he is overtaken by B?

Ans. 50 hours, and 70 miles.

11. A cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute? Ans. 22, 7 and 12 gallons.

12. Divide 84 into two such parts, that if one-half of the less be subtracted from the greater, and one-eighth of the greater be subtracted from the less, the remainders shall be equal. Ans. 48 and 36.

13. A thief is running away from a certain place at the rate of 15 miles in 4 hours. 24 hours later, a constable is making after him, at the rate of 20 miles in 3 hours. In how many hours will the thief be taken?

Ans. $30\frac{6}{7}$ hours.

14. There are two numbers in the proportion of 5 to 4;

but if each be increased by twenty, the result will be as 9 to 8. What are the numbers? ANS. 25 and 20.

15. It is required to divide the number 91 into two such parts, that the greater being divided by their difference, the quotient may be 7. ANS. The parts are 49 and 42.

16. Two boys, standing with bows and arrows on the bank of a river, undertook to shoot across it; the arrow of the first boy fell 10 yards short of the opposite bank, and that of the second fell 10 yards beyond it; now, it was found that the first boy shot only nine-elevenths as far as the second. What was the breadth of the river?

ANS. 100 yards.

17. The joint stock of two partners, whose particular shares differed by £40, was to the share of the less as 14 to 5. Required the shares.

ANS. £90 and £50, respectively.

18. Two friends met a horse-dealer leading a horse, which they resolved to buy jointly. When they had agreed as to the price, they found that the one was able to pay only the fifth part, and the other the seventh part; this they put together, and paid the seller therewith, on account, $48. What was the price of the horse? ANS. $140.

19. Divide the number 46 into two unequal parts, so that when one is divided by 7 and the other by 3, the quotients together may amount to 10. What are these parts?

ANS. 28 and 18.

20. An arithmetician desires his scholars to find a number, which he has in his mind, from the following data: If, says he, you multiply the number by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, you will obtain this same number. What number then, is it? ANS. 54.

21. A master hired a journeyman, and promised him 8 shillings for each day that he worked for him; but if he worked anywhere else, then the journeyman must pay him 5 shillings daily for his board. At the expiration of 50 days they settle, and the journeyman receives 36 dollars and 2 shillings. How many days had he worked for his master?

ANS. 36 days.

22. A, B and C, can perform a piece of work in 5 days. A alone can do it in 12 days, and B in 15. In what time could C accomplish it? ANS. 20 days.

23. A and B commenced trade, A with twice as much money as B. A gained $20, and B lost $10; then the difference between A's and B's money was $70. How much did each begin with? ANS. A $80, B $40.

24. A poulterer had a certain number of geese, and twice as many turkeys; after having sold 10 geese and bought 30 turkeys, he found that if he subtracted three-fifths of his number of geese from his number of turkeys, the remainder would be the same as if he subtracted eight-fifteenths of his number of turkeys from 4 times his number of geese. How many of each had he at first?

ANS. 60 geese, and 120 turkeys.

SECTION V.—ALGEBRAIC OPERATIONS.

14. Every number written in algebraic language is called an *algebraic quantity*, or an *algebraic expression*. Thus, $7 + 5$ is an algebraic quantity, or the algebraic expression for the sum of 7 and 5.

In like manner, $3a$ is the algebraic expression for three times the number a; $5\,a\,b$ is the algebraic expression for five times a multiplied by b; $3\,a + 2\,b - c$ is the algebraic expression for three times a, increased by $2\,b$, and diminished by c.

A quantity consisting of one term only is called a simple quantity, or *monomial;* thus, $3\,a$, $5\,b$, are simple quantities, or monomials.

If the quantity consists of two terms, it is called a *binomial;* thus, $a + b$, $x - y$, are binomials.

If the quantity consists of three terms, it is called a *trinomial.* In general, if a quantity consists of more than one term, it is called a *polynomial.*

The product of two quantities, a and b, we have seen, is expressed by writing these quantities one after the other, thus: $a\,b$. In like manner, the product of three quantities, a, b and c, is expressed thus, $a\,b\,c$; and so on, for any number of quantities.

When two or more quantities are multiplied together, forming a product, these quantities are called the *factors* of the product. Thus a, b, c, are factors of the product $a\,b\,c$.

When the factors of a product are all equal, as in the case of the products $a\,a$, $a\,a\,a$, &c., the product may be more concisely expressed. In order to this, we write one of the factors only; and at the right of this, and a little above it, we place a figure to denote the number of times it is repeated as a factor. Thus the product $a\,a$, in which a occurs *twice* as a factor, we write a^2. The product $a\,a\,a$, in which a occurs *three* times as a factor, we write a^3, and so on.

When a letter is repeated as a factor, the product is called a *power* of that letter, and the degree of the power is indicated by the figure employed to denote the number of times it occurs as a factor; thus, a^2 is called the *second* power of a, and is read *a second* power. In like manner, a^3 is read *a third* power, and so on.

The figure which denotes the power is called the exponent of the power; thus, in a^3, a^5, the 3 and 5 are the exponents of the power to which a is said to be raised.

The exponent should be carefully distinguished from the coefficient, which is always placed at the left of the quantity, and on the same line with it. Thus five times a seventh power is written $5\,a^7$.

A letter which has no exponent is regarded as having unity for an exponent, in the same manner that a letter without a coefficient is regarded as having unity for a coefficient.

Ex. 1. What is the algebraic expression for three times the square of a multiplied by the cube of b?

Ex. 2. Write seven times the cube of a multiplied by b, diminished by the square of c multiplied by d fifth power.

Ex. 3. Write five times the seventh power of a multiplied by b, and divided by the square of c.

Ex. 4. Write three times the third power of a, diminished by the square of c, and divided by d.

Ex. 5. Write five-fourths of a square, diminished by b square.

Quantities are said to be similar when the literal factors of which they are composed are the same; thus, $3\,a\,b$, $5\,a\,b$, are similar quantities. In like manner, $7\,a^2\,b$, $4\,a^2\,b$, are similar quantities. The quantities $7\,a^3\,b$, $4\,a^2\,b$, are dissimilar, since the literal factors are not the same in each, the letter a occurring once more as a factor in the former than it does in the latter.

The *degree* of a term is the number of its literal factors. It is found by adding together the exponents of all the letters. Thus $7\,a^3\,b^2\,c$ is a term of the sixth degree.

A polynomial is said to be *homogeneous* when all its terms are of the same degree. Thus, $5a^2 - 3ab + b^2$ is a homogeneous polynomial of the second degree.

The numerical value of an algebraic expression is the number which results from giving particular values to the letters which enter into the expression, and performing the operations indicated by the algebraic signs. Thus the numerical value of $3a^2$, when a is equal to 5, is 75; for the square of 5 is 25, and three times 25 is 75. If a is equal to 7, then $3a^2$ is equal to 147. In like manner, if a is equal to 4 and b is equal to 3, the numerical value of $a^3 - b^2$ will be 55.

Ex. 1. Find the numerical value of the following expressions when $a = 3$, $b = 2$ and $c = 5$.

1. $a - b + c$. Ans. 6.
2. $a^2 + ab - c$. Ans. 10.
3. $a^2 + 2ab + b^2$. Ans. 25.
4. $ac - ab + bc$. Ans. 19.

Ex. 2. What will be the value of the same expressions, when $a = 5$, $b = 3$, $c = 2$?

15. From the preceding examples, we have occasion, it is evident, to perform in Algebra operations analogous to addition, subtraction, multiplication and division, in Arithmetic, and which go by the same name. We now proceed to a more full explanation of these operations.

ADDITION AND SUBTRACTION OF MONOMIALS.

16. 1. Let it be required to add together the monomials ab, bc, cd. The quantities here being dissimilar, the addition can only be expressed by aid of the sign $+$; thus,

$$ab + bc + cd.$$

2. Let it be required to add $4\,a^3\,b$, $7\,a^3\,b$ and $5\,a^3\,b$. Here the quantities are similar; and the $a^3\,b$, it is evident, is taken 4 times, 7 times and 5 times. On the whole, then it is taken 16 times, and we shall have

$$4\,a^3\,b + 7\,a^3\,b + 5\,a^3\,b = 16\,a^3\,b.$$

In general, if the quantities are similar, the result is obtained by placing the sum of the coefficients before the literal factors.

3. Add the monomials $3\,a^3\,b^2\,c$, $5\,a^3\,b^2\,c$, $9\,a^3\,b^2\,c$, and $12\,a^3\,b^2\,c$. Ans. $29\,a^3\,b^2\,c$.

4. Add the monomials $5\,x^2\,y$, $7\,x^2\,y$, $8\,x^2\,y$ and $20\,x^2\,y$. Ans. $40\,x^2\,y$.

5. Add the monomials $10\,a^5\,b^2\,c$, $7\,a^3\,b\,c$, $8\,a^5\,b^2\,c$, $3\,a^3\,b\,c$ and $a\,b\,c$. Ans. $18\,a^5\,b^2\,c + 10\,a^3\,b\,c + a\,b\,c$.

17. 1. Let it be required next to subtract $2\,a\,b$ from $3\,c\,d$. The quantities being dissimilar, the subtraction can only be indicated by the sign —, thus $3\,c\,d - 2\,a\,b$.

2. Let it be required next to subtract $3\,a^2\,b$ from $5\,a^2\,b$. Here the quantities are similar, and $5\,a^2\,b$ less $3\,a^2\,b$ is equal, it is evident, to $2\,a^2\,b$.

In general, if the quantities are similar, the subtraction is effected by taking the difference of the coefficients and placing it before the literal factors, as the coefficient of the remainder.

POSITIVE AND NEGATIVE QUANTITIES.

18. Let it next be required to subtract $7\,a^2\,b$ from $5\,a^2\,b$. The subtraction here cannot be performed, since we are required to take a greater quantity from a less, which is impossible. From $5\,a^2\,b$, however, we may take $5\,a^2\,b$, a quantity equal to itself, and then there will be left $2\,a^2\,b$

still to be subtracted. To denote this result, we place the sign — before the $2a^2b$, thus, $-2a^2b$; and we say that if $7a^2b$ be taken from $5a^2b$, the remainder will be $-2a^2b$; that is, there will be left $2a^2b$ still to be subtracted.

A quantity with the sign + before it is called a *positive quantity*. In Algebra the sign + is always understood before quantities which have no sign before them.

A quantity with the sign — before it is called a *negative quantity*; thus, $-5a^3b$, $-9a^4b^2$, are negative quantities. Negative quantities arise from the attempt to subtract a greater quantity from a less. They are, therefore, regarded as quantities still remaining to be subtracted.

19. In Algebra, however, the signs + and — are employed, in general, to denote quantities in precisely opposite circumstances to each other, or which tend to produce opposite effects. Thus, in estimating the journeyings of a traveller, if we designate the distances he goes east by the sign +, we shall designate those he goes west, or in the opposite direction, by the sign —.

In like manner, in estimating the amount of a man's property, his assets, such as Bank Stock, Real Estate, &c., we regard as positive, and his debts, or whatever tends to diminish his estate, as negative. The former we mark with the sign +, the latter with the sign —.

1. A surveyor runs several courses: on the first he makes 7 rods west; on the second 9, and on the third 13 rods east; on the fourth 8, and on the fifth 10 rods west. On which side is he of the point from which he starts, and what is the distance on an east and west line?

Regarding distance east as positive, the aggregate of the eastings and westings will be expressed thus:

$$-7+9+13-8-10=22-25=-3.$$

The sign — in the result shows that the surveyor is west of his starting-point, and the 3 shows that the distance on an east and west line is three rods.

If, in this example, we regard distance west as positive, distance east will be negative, and the sum or aggregate of the distances will be expressed thus:

$$7 - 9 - 13 + 8 + 10 = 25 - 22 = +3.$$

The result is the same as before; the sign +, in this case, indicating that the surveyor is west of his first position, and the 3 indicating the distance west.

Positive and negative quantities, it is evident, differ only in the sense or direction in which they are taken.

2. A gentleman has Real Estate valued at $3000, and Bank Stock valued at $2500. He owes one creditor $700 and another $1900. What is he worth?

Ans. $2900.

3. A merchant has in goods $5000, Real Estate $3000, Rail Road Stock $2500. But he owes one creditor $4000, another $6000, and there is an incumbrance on his Real Estate of $1500. What is he worth?

Ans. — $1000.

The sign — here shows that the merchant, instead of being worth anything, is, on a settlement of his affairs, a thousand dollars in debt.

A negative quantity is sometimes said to be less than nothing. It is so in this sense only, that an equal positive quantity must be added to it to make the result equal to 0. In the last example, the property of the merchant might be stated as $1000 less than nothing. That is, he must, in some form, acquire $1000, in order that he may be just out of debt, on the one hand owing nothing, and on the other possessing no property.

ADDITION AND SUBTRACTION OF MONOMIALS, SOME OF WHICH ARE POSITIVE AND OTHERS NEGATIVE.

20. The effects of a merchant are represented by the monomials $4a$, $5a$, and $7b$, and his debts by the monomials $3a$, $5b$. What is the amount of his property?

From what has been said, his effects, it is evident, should be marked with the sign $+$, and his debts with the sign $-$; thus, $-3a$, $-5b$. The aggregate of his property will then be expressed thus:

$$4a + 5a + 7b - 3a - 5b;$$

or, reducing, it becomes $6a + 2b$.

In obtaining, in this manner, the value on the whole, or aggregate of the merchant's property, the debts, with their proper sign, may be considered as added with the effects. And, in this point of view, it will be seen that *the addition of a negative quantity is the same as the subtraction of an equal positive quantity.*

From what has been done, we have, it is evident, the following rule for the addition of monomials affected with the signs $+$ and $-$, viz.: *Write the quantities one after the other, with their proper signs, taking care to reduce terms which are similar.*

Ex. 1. Add the monomials $4a^2b$, $5ab^2$, $7a^2b$, $-7ab^2$, and $-3a^2b$. ANS. $8a^2b - 2ab^2$.

Ex. 2. Add the monomials $9a^3b^2$, $-7ab$, $4a^3b^2$, $5cd$, $-3ab$, $7a^3b^2$, $-4cd$ and $10ab$.

ANS. $20a^3b^2 + cd$.

Ex. 3. Add the monomials $3a$, $-5a$, $7a$, $2b$, $-c$ and $4b$. ANS. $5a + 6b - c$.

Ex. 4. Add the monomials, $-7b$, $3a$, $4b$, $-2a$, $3b$, $-3a$ and $2a$. ANS. 0.

21. A has property worth \$2000; B is \$1000 in debt. What is the difference of their property? It is evident that the difference is \$3000. For, in order that B may be as well off as A in respect to property, he must first acquire \$1000 to pay his debt, and, in addition to this, \$2000 more. A's property is + \$2000, B's — \$1000; the latter subtracted from the former, in order to obtain the difference, gives, therefore, \$3000. From this it will be seen that *the subtraction of a negative quantity is the same as the addition of an equal positive quantity.*

The rule for subtraction will then be, *Change the sign of the quantity to be subtracted, and proceed as in addition.*

Ex. 1. Subtract $-5a^3b^2$ from $7a^3b^2$.

Ans. $12a^3b^2$.

Ex. 2. Subtract $4ab$ from $-7ab$.

Ans. $-11ab$.

Ex. 3. Subtract $-3a^2b$ from $-9a^2b$.

Ans. $-6a^2b$.

Ex. 4. Subtract $-7cd$ from $-4cd$. Ans. $3cd$.

Ex. 5. On a thermometer the graduation, commencing at a point marked 0, extends above and below this point. The degrees above are marked +; those below are marked —. If, on the 20th of January, the thermometer stood at 15° below 0, and on the day following 7° above, which was the colder day, and by how much?

Ex. 6. At sunrise, on the 22d of December, the reading of the thermometer at Portland was — 22°, and on the same day at Boston it was + 3°. What was the difference of temperature between the two places?

Ex. 7. A ship, by observation, finds her longitude to be + 40°; thirty days after, it is — 20°. What is the difference of longitude she has made?

Ex. 8. The latitude of New Orleans is 30° north; that of Buenos Ayres is 34° south. What is the difference of latitude between the two places?

Ex. 9. A has property valued at \$5000; B is \$3450 in debt. What is the difference in the amount of their property? What are they both together worth?

REDUCTION OF TERMS.

22. The process by which several similar terms are united in one is called *reduction*, or the reduction of similar terms. In order that a result may be left in the most simple state possible, all similar terms should be reduced to one term.

Find the most simple form of the following expressions:

1. $a + 2b - 3c + 3a - 5b$.
2. $6a^2b - 5c^2d + bc - 2a^2b + 3c^2d - 7bc$.
3. $7ab - 4cd + 2ab + 3cd - 9ab$.
4. $5x^2 + 3xy - 2x^2 - 7xy$.

ADDITION OF POLYNOMIALS.

23. Let it be required next to add the polynomials $a + b - c$ and $b - 2c$. The first polynomial is the aggregate of the monomials a, b and $-c$; the second is the aggregate of the monomials b and $-2c$; the sum of the two polynomials will be, therefore, the aggregate of the monomials a, b, $-c$, b, $-2c$. We shall have, then, for the sum required,

$$a + b - c + b - 2c;$$

or, reducing, $$a + 2b - 3c.$$

The result is obtained by writing the polynomials one after the other with their proper signs, and uniting similar terms.

24. The value of a polynomial will be the same in whatever order the terms are written, provided the proper signs are preserved; thus, $17 - 8$, and $-8 + 17$ have both the same value, viz., 9. In like manner, $3cx - 9a - 5b$ is the same with $-9a + 3cx - 5b$, or with $-9a - 5b + 3cx$. Taking advantage of this remark, we may arrange the proposed polynomials so that their similar terms shall fall under each other, and thus facilitate their addition.

Let it be required, for example, to add the polynomials $3a - 2b + 4cx$, $7cx - 3b + 8a$, $3cx - 9a - 5b$. Writing the proposed so that the similar terms shall stand under each other, they may be prepared for addition thus:

$$\begin{array}{r} 3a - 2b + 4cx \\ 8a - 3b + 7cx \\ -9a - 5b + 3cx \\ \hline 2a - 10b + 14cx \end{array}$$

Adding and reducing the terms which stand under each other, the result will evidently be as written above,

$$2a - 10b + 14cx.$$

25. Addition in Algebra consists, it is evident, in finding the most simple quantity equivalent to the aggregate or sum of several different quantities.

From what has been done, we have the following general rule for the addition of algebraic quantities:

1°. *Write the quantities to be added so that the similar terms, each with its proper sign, shall fall under each other in vertical columns.*

2°. *Reduce the similar terms, and annex to the results the terms which are not similar, giving to each term its proper sign.*

EXAMPLES.

1. Add the following polynomials: $6a + 5y - 6bc$, $7bc + 4a - 6y$, $12y + 7a - 3bc$, and $15y + 2bc - 4a$. ANS. $13a + 26y$.

2. Add the following polynomials: $8a + b$, $c - b + 2a$, $5b - 3a + 2d$, $3d - 6b - 3c$, $7c - 2d - 5a$.
ANS. $2a - b + 5c + 3d$.

3. Add $7x - 5y - 9bc$, $3bc - 5x - 4y$, $3x + 7y - 4bc$, $3bc - 4x - 10y$.
ANS. $x - 12y - 7bc$.

4. Add $5a + 4b - 3cx$, $2cx + 7a - 3b$, $-3a - 7b - 3cx$, and $5b + 9a + 12cx$.
ANS. $18a - b + 8cx$.

5. Add $ax - 4ab + bd$, $3bd - 2ax + ab$, $7ab - 2ax - bd$, and $5ab - 3ax + 12bd$.
ANS. $9ab - 6ax + 15bd$.

6. Add

$$\begin{array}{r} 7x - 6y + 5z + 3 - g \\ -x - 3y \quad\quad - 8 - g \\ -x + y - 3z - 1 + 7g \\ -2x + 3y + 3z - 1 - g \\ x + 8y - 5z + 9 + g \end{array}$$

ANS. $4x + 3y + 2 + 5g$.

7. What will be the numerical value of the expressions in the example first proposed, if we suppose $a = 7$, $b = 5$ $c = 3$, $x = 2$?

$$\begin{array}{rr} 3a - 2b + 4cx = & 35 \\ 8a - 3b + 7cx = & 83 \\ -9a - 5b + 3cx = & -70 \\ \hline 2a - 10b + 14cx = & 48 \end{array}$$

8. What will be the numerical value of the sum of the polynomials in the same example, if we suppose $a = 5$, $b = 2$, $c = 3$, $x = 7$? ANS. 284.

In both cases it will be seen that the numerical value of the result $2a - 10b + 14cx$, is the same with that of the sum of the numerical value of the polynomials taken separately. Let the learner make similar substitutions in the other examples.

SUBTRACTION OF POLYNOMIALS.

26. Let it be required next to subtract $4ax + 3bc - 2by$ from $7ax - 3bc + 8by$. The subtraction, it is evident, will be effected, if from the minuend we subtract each of the terms of the subtrahend, regard being had to their signs. Performing the operation, we have

$$7ax - 3bc + 8by - 4ax - 3bc + 2by;$$

or, reducing,

$$3ax - 6bc + 10by.$$

The result is obtained by writing the quantity to be subtracted, with a change of its signs, after that from which it is to be taken, and then uniting similar terms.

As in addition, the operation will be facilitated by writing the polynomial to be subtracted directly under that from which it is to be taken, and so disposing the terms that those which are similar shall fall under each other.

Ex. From $8ax - 3by + 4dx$ take $3dx - 5ax + 4by$. The polynomials may be arranged and the work performed as follows:

$$\begin{array}{r} 8ax - 3by + 4dx \\ -5ax + 4by + 3dx. \\ \hline \end{array}$$

ANS. $13ax - 7by + dx$.

27. Subtraction in Algebra consists in finding the most simple expression for the difference between two algebraic quantities.

From what has been done, we have the following general rule for the subtraction of algebraic quantities:

1°. *Write the quantity to be subtracted under that from which it is to be taken, so that the similar terms, if there are any, shall fall under each other.*

2°. *Change all the signs in the quantity to be subtracted, or suppose them to be changed, and then proceed as in addition.*

EXAMPLES.

1. From $17c^3 - d + 14a^3 - 4b^2$ take $3a^3 + 2b^2 + 15c^3 - d$. Ans. $11a^3 - 6b^2 + 2c^3$.

2. From $13a - 2b + 9c - 3d$ take $9c + 8a + 12 - 6b - 10d$. Ans. $5a + 4b + 7d - 12$.

3. From $9a^4b^5 - 9a^3b^5 + 3$ take $d - 3a - 9a^3b^5 + 9a^2b^4$. Ans. $3a + 3 - d$.

4. From $-14b + 3c - 27d + 3 - 5g$ take $7a - 5c - 8d + 3b - 12 + 7g$.
Ans. $-7a - 17b + 8c - 19d + 15 - 12g$.

28. Subtraction, as we have seen, is indicated by writing the quantity to be subtracted within a parenthesis, and placing the sign — before it; thus, $a - b + c$ subtracted from $c + d$ is written

$$c + d - (a - b + c).$$

If we now perform the subtraction, the result will be

$$c + d - a + b - c;$$

or, reducing, $d - a + b$.

If, then, in an expression of the form

$$c + d - (a - b + c),$$

we remove the parenthesis, the signs of all the terms enclosed within it must be changed.

Ex. 1. What is the expression equivalent to,

$$5a^2b - 3cd - (2a^2b^2 - 4cd^2),$$

the parenthesis being removed?

Ans. $5a^2b - 3cd - 2a^2b^2 + 4cd^2$.

Ex. 2. Reduce to its simplest form the following expres sion:

$$8a^3b + 4a^2b^2 - 3ab^3 - (7a^3b + 4a^2b^2 - 5ab^3)$$

Ans. $a^3b + 2ab^3$.

Ex. 3. What more simple quantity is equivalent to the following?

$$7cd - 3ab + 4a^2 - (3cd + 2ab + 4a^2).$$

Ans. $4cd - 5ab$.

29. A process the reverse of the preceding is sometimes required. For example, what will be the equivalent expression for $a + b - c + d$, supposing the three last terms placed within a parenthesis, with the sign — before it?

Ans. $a - (-b + c - d)$.

If we perform the operations indicated in this last expression, it will return, evidently, to the former.

What will be the equivalent expressions for the following polynomials, supposing the last two terms in each placed within a parenthesis, with the negative sign before it?

1. $a^2 - b^2 + c^2$.
2. $3a^2b - 4a^3b^2 + 2a^4b^3 - a^5b^4$.
3. $a^3b - 4a^2b^2 + 7a^3b^3$.

MULTIPLICATION OF ALGEBRAIC QUANTITIES.

MONOMIALS.

30. 1. Let it be required to multiply $5ab$ by $3cd$. The product, we have seen, will be expressed thus:

$$5ab \times 3cd.$$

But since it is indifferent what order is observed among the factors, the product may be written thus:

$$5 \times 3\, abcd;$$

or, reducing, $15\, abcd$.

2. Let it be required next to multiply $7\, a^3b^2$ by $4\, a^2b$. The result, from what has been done, will be

$$28\, a^3b^2a^2b.$$

In this result, a occurs, it is evident, three times as a factor, and also twice as a factor; it, on the whole, occurs, therefore, five times as a factor. In like manner, b occurring twice and once as a factor, occurs, on the whole, three times as a factor; the result may, therefore, be expressed more concisely thus: $28\, a^5b^3$.

We have, then, the following rule for the multiplication of monomials, viz.:

1°. *Multiply the coefficients, as in arithmetic.*

2°. *To this product annex all the letters in each of the factors, observing to give to each letter an exponent equal to the sum of its exponents in the factors.*

31. In the preceding examples the quantities to be multiplied are each positive; that is, each is supposed to be affected with the sign +. Let us now examine the cases in which one or both of the quantities are negative; that is, affected with the sign —.

The algebraic signs + and —, it will be recollected, indicate merely operations upon quantities, or the sense or direction in which certain quantities are to be taken. They do not at all affect the nature of the quantities before which they are placed. The number 5, for example, is in itself in every respect the same, whatever sign, whether + or —, is placed before it. These signs simply indicate its relation to other quantities. The one shows that it is intended to be added to some other quantity, the other that it is to be subtracted from it. The one shows that the magnitude represented by it lies in one direction, the other that it lies in the opposite; or, that in some way the quantities which it represents are in opposite circumstances in respect to each other.

The multiplier, independently of its sign, indicates only the number of times the multiplicand is to be repeated. The sign placed before the multiplier indicates an additional circumstance. It shows by what manner of operation, or in what sense or direction, the multiplicand is to be repeated. If the sign $+$ is placed before the multiplier, it indicates that the multiplicand should be repeated *with its sign*, whether $+$ or $-$, as many times as there are units in the multiplier. If the sign $-$ is placed before the multiplier, it indicates that the multiplicand should be repeated in a manner the *reverse* of its sign, whether it be $+$ or $-$, as many times as there are units in the multiplier.

1. Let it now be proposed to multiply $-a$ by b. Here the a is to be repeated, with its sign, as many times as there are units in b; that is, b times. The product will be, therefore, $-a\,b$.

2. Let it next be required to multiply a by $-b$. Here a is to be repeated, in a manner the reverse of its sign, as many times as there are units in b, or b times. The product will be, therefore, $-a\,b$.

3. Again, let it be required to multiply $-a$ by $-b$. Here $-a$ is to be repeated also, in a manner the reverse of its sign, a number of times denoted by b. The product will, therefore, be $a\,b$.

We shall have, then, the following rule for the signs: *If two quantities have like signs, that is, both* $+$ *or both* $-$, *their product will have the sign* $+$; *but if the quantities* *have unlike signs, that is, one + or the other —, the product will have the sign —.*

EXAMPLES.

Multiply 1. a^3b^2 by $a^4b^5c^2$. Ans. $a^7b^7c^2$.

2. $-5\,a^4b^3c$ by $7\,a^2b^7cd$.

Ans. $-35\,a^6b^{10}c^2d$.

3. $-9\,ab^5c^2d$ by $-3\,a^4b^2$.

Ans. $27\,a^5b^7c^2d$.

POLYNOMIALS.

31. We pass next to the multiplication of polynomials. The multiplication of two polynomials, $a + b$, $c + d$, for example, is indicated by enclosing each of the polynomials in a parenthesis, and writing them one after the other, either with or without the sign of multiplication, thus:

$$(a + b) \times (c + d), \text{ or } (a + b)\,(c + d).$$

1. Multiply $a + b - c$ by m. The product, it is evident, will be $am + bm - cm$.

2. Multiply $a + b - c$ by $d + e$. Here the multiplicand is to be taken $d + e$ times, or, which is the same thing, d times $+ e$ times. We shall obtain the product, therefore, by multiplying $a + b - c$ first by d and then by e, and adding the partial products thus obtained. The operation may be performed as follows:

$$\begin{array}{l} a + b - c \\ d + e \\ \hline ad + bd - cd \\ \qquad\quad ae + be - ce \\ \hline ad + bd - cd + ae + be - ce \end{array}$$

3. Multiply $a + b - c$ by $d - e$. Here the multipli

cand is to be taken $d - e$ times, or d times diminished by e times. We shall obtain the product, therefore, if from d times the multiplicand we subtract e times the multiplicand. The products, d times the multiplicand, e times the multiplicand, will be the same as above; writing the latter with a change of sign, as it is to be subtracted, the operation will be as follows:

$$\begin{array}{l} a+b-c \\ d-e \\ \hline ad+bd-cd \\ \qquad -ae-be+ce \\ \hline ad+bd-cd-ae-be+ce \end{array}$$

If we examine the examples above with attention, it will be seen that the rule for the signs is the same, whether the terms which are multiplied together stand alone, or are united, so as to form polynomials.

Indeed, the investigation of the rule for polynomials comprehends that for monomials also. The form of the product, in the two preceding examples, does not, it is evident, at all depend upon the particular values we may assign to the letters a, b, c, and d. It will be the same, whatever those values may be. Suppose, then, that in the last example a, d, and c, are each equal to 0. The product will be reduced to $-be$, and we shall have $b \times -e = -be$. If we suppose a, b, and d, each equal to 0, the product will be reduced to ce, and we have $-c \times -e = ce$.

The rule for the signs may then be briefly stated thus: *Like signs produce* $+$, *unlike signs produce* $-$.

32. In the multiplication of polynomials, partial products may arise, which are similar. These should be reduced, so that the result may appear in the most simple form.

A polynomial is said to be arranged with reference to some letter, when its terms are written in order according to the powers of that letter, beginning either with the highest or the lowest power. The polynomial $a^2b^3 + a^3b - ab^4 + a^4b^2$, arranged with reference to the powers of the letter a, will stand thus: $a^4b^2 + a^3b + a^2b^3 - ab^4$; or thus, $-ab^4 + a^2b^3 + a^3b + a^4b^2$. In the first case, it is said to be arranged in descending powers of the letter a; in the second, in ascending powers of the letter a. The letter a, with reference to which the arrangement is made, is called the *principal* letter.

To facilitate the multiplication of polynomials, the quantities to be multiplied should each be arranged according to the powers of the same letter, and the partial products should be so disposed that similar terms may fall under each other.

33. The following example illustrates fully the course to be pursued in the multiplication of polynomials.

Let it be proposed to multiply the polynomial

$$3b^2a - b^3 + a^3 - 3ba^2$$

by

$$-2ba + a^2 - 4b^2.$$

Arranging with reference to the descending powers of the letter a, the work will be as follows:

$$\begin{array}{llllll}
a^3 - 3ba^2 + 3b^2a - b^3 & & & & & \\
a^2 - 2ba \;\; - 4b^2 & & & & & \\
\hline
a^5 & -3ba^4 & +3b^2a^3 & -b^3a^2 & & \\
 & -2ba^4 & +6b^2a^3 & -6b^3a^2 & +2b^4a & \\
 & & -4b^2a^3 & +12b^3a^2 & -12b^4a & +4b^5 \\
\hline
a^5 & -5ba^4 & +5b^2a^3 & +5b^3a^2 & -10b^4a & +4b^5
\end{array}$$

From what has been done, we have the following rule for the multiplication of polynomials:

1°. *Arrange the proposed polynomials according to the powers of the same letter.*

2°. *Multiply each term of the multiplicand by each term of the multiplier, observing that if the terms are affected each with the same sign, the product should have the sign +; but if with different signs, the product should have the sign —.*

3°. *Add together the partial products thus obtained, taking care to unite in one terms which are similar.*

EXAMPLES.

1. Multiply $x^3 - 3x^2y + 3xy^2 - y^3$
by $x^2 - 2xy + y^2$.
ANS. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

2. Multiply $x^2 + 2xy + y^2$
by $x^2 - 2xy + y^2$.
ANS. $x^4 - 2x^2y^2 + y^4$.

3. Multiply $3a^4 - 7a^3b + b^3$
by $2a^2 - 4ab$.
ANS. $6a^6 - 26a^5b + 28a^4b^2 + 2a^2b^3 - 4ab^4$.

4. Multiply $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$
by $a - b$. ANS. $a^6 - b^6$.

5. Find the product of the four factors,
$a - 4x$, $a - x$, $a + x$ and $a + 4x$.
ANS. $a^4 - 17a^2x^2 + 16x^4$.

DIVISION OF ALGEBRAIC QUANTITIES.

MONOMIALS.

34. Division, in Algebra, as in Arithmetic, is the reverse of multiplication. In the latter, two factors of a product are given to find the product; in the former, the product and one of the factors are given to find the other factor. The divisor and quotient, multiplied together, should, therefore, reproduce the dividend.

This being the case, it is evident, that if we can discover in the dividend the factors of the divisor, the quotient will be obtained by striking these factors out of the dividend. Thus, let it be required to divide $a\ b\ c\ d$ by $b\ d$. Striking out of the dividend the factors b and d of the divisor, we have $a\ c$ left for the quotient. This is evident; for, if we multiply $a\ c$ by $b\ d$, the result will be $a\ c\ b\ d$, or, which is the same thing, $a\ b\ c\ d$, the dividend.

2. If the divisor and dividend have coefficients, the coefficient of the dividend must be divided by the coefficient of the divisor; the result will be the coefficient of the quotient. Thus, the quotient of $18\ b\ c\ d$, divided by $3\ b\ d$, will be $6\ c$.

Ex. 1. Divide $9\ a\ b\ c\ x$ by $3\ x$. Ans. $3\ a\ b\ c$.

2. Divide $25\ x\ y\ z$ by $5\ x\ z$. Ans. $5\ y$.

3. Divide $14\ a\ d\ b\ x\ y$ by $2\ a\ b\ y$. Ans. $7\ d\ x$.

3. Let it be required next to divide $a^5\ b^2\ c$ by $a^3\ c$. The dividend may be decomposed, it is evident, into the factors $a^3\ a^2\ b^2\ c$. Striking out from this last the factors of the divisor, we obtain the quotient, $a^2\ b^2$. The effect would be the same, if, in respect to letters affected with exponents, the exponent of the divisor were subtracted from that of the dividend. Indeed, this should be the case, since in multiplication the exponents of the same letter are added, in order to produce the product. Thus, $a^3\ c$, multiplied by $a^2\ b^2$, gives $a^5\ b^2\ c$ for the product.

If the exponent of the divisor is the same as that of the dividend, the result will have 0 for its exponent; thus,

$$\frac{a^2}{a^2} = a^0.$$

But $\frac{a^2}{a^2}$, it is evident, is equal to 1, since any quantity divided by itself, will give unity for a quotient. *The expres-*

6*

sion a^0, therefore, and, in general, any quantity with 0 for an exponent, may be regarded as equivalent to unity.

4. Let it be required next to divide $-8a^7b^3c^2$ by $4a^3b^2c$ From what has been said, the quotient of $8a^7b^3c^2$ by $4a^3b^2c$ is $2a^4bc$. But since the dividend is affected with the sign $-$, what sign shall we give to the quotient? The sign, it is evident, should be $-$, since the quotient, multiplied by the divisor, should reproduce the dividend, and $4a^3b^2c$ multiplied by $-2a^4bc$ gives $-8a^7b^3c^2$. In like manner, if $-a^5bc$ be divided by $-a^3b$, the quotient will be a^2c; since a^2c, multiplied by $-a^3b$, gives $-a^5bc$.

From what has been done, the following rule for the division of monomials will be readily inferred, viz.:

1°. *Divide the coefficient of the dividend by the coefficient of the divisor.*

2°. *Strike out from the dividend the letters common to it and the divisor, when the exponents in each are the same; but when the exponents are not the same, subtract the exponent of the letter in the divisor from that of the letter in the dividend, and write the letter in the quotient with an exponent equal to the remainder.*

3°. *Write in the quotient, with their respective exponents, the letters in the dividend not found in the divisor.*

4°. *If the divisor and dividend have like signs, the quotient should have the sign $+$; if they have different signs, the quotient should have the sign $-$.*

EXAMPLES.

1. Divide $8a^3b^3c^2d$ by $4ab^2c^2$. ANS. $2a^2bd$.
2. Divide $75a^5b^3cd^3$ by $25a^2bd$. ANS. $5a^3b^2cd^2$.
3. Divide $120a^7b^4m^2n$ by $-10a^3b^2m$.
ANS. $-12a^4b^2mn$

4. Divide $-63 a^9 b^7 x^4 y^2$ by $-9 a^5 b^3 x^2 y^2$.

Ans. $7 a^4 b^4 x^2$.

5. Divide $9 m^2 n^2 x^3 y^5$ by $3 m n^2 x^2 y^3$.
6. Divide $150 a^{10} b^7 x^5 y^2 z$ by $-50 a^7 b^5 x^2 y$.
7. Divide $-84 x^7 y^3 z$ by $7 x^5 y^2 z$.
8. Divide $-906 m^2 n^2 x^3 y^5$ by $-6 m^2 n^2 y^2$.

5. Let it be required next to divide $7 a^2 b$ by $3 c d$. In this case the division cannot be performed. It may be indicated, however, by writing the divisor beneath the dividend, in the form of a fraction, thus:

$$\frac{7 a^2 b}{3 c d}.$$

Expressions of this form are called *algebraic fractions.*

POLYNOMIALS.

35. Let it be required to divide $a^4 b - a^3 c + a^2 b$ by a^2. Reversing the process of multiplication, we divide each term of the proposed by a^2, which gives $a^2 b - a c + b$ for the quotient. This, it is evident, is the true quotient, since if we multiply it by a^2 we reproduce the polynomial required to be divided.

Ex. 1. Divide $8 x^2 - 4 x^3 + 2 x y^2$ by $2 x$.

Ans. $4 x - 2 x^2 + y^2$.

Ex. 2. Divide $14 a^3 b - 21 a^2 b^2 + 14 a b^3$ by $7 a b$.

Ans. $2 a^2 - 3 a b + 2 b^2$.

Ex. 3. Divide $-25 a x^4 + 75 a^2 x^5 - 100 a^3 x^6$ by $25 a x^4$. Ans. $-1 + 3 a x - 4 a^2 x^2$.

Ex. 4. Divide $9 a^5 b - 27 a^4 b^2 + 36 a^3 b^3$ by $-9 a^2 b$.

From what has been done, we have the following rule for the division of polynomials by a monomial: *Divide each term of the polynomial by the monomial, observing to give to each partial quotient, as we proceed, the proper sign.*

36. Let the purposed next be two polynomials. In order to determine how to proceed in this case, let us notice more particularly the manner in which the product of two poly nomial factors is formed.

Let the proposed be $2a^3 - 5a^2b + ab^2$ and $3a^2 + 2ab$ Performing the multiplication, the operation will stand thus

$$\begin{array}{l} 2a^3 - 5a^2b + ab^2 \\ 3a^2 + 2ab \\ \hline 6a^5 - 15a^4b + 3a^3b^2 \\ \qquad\quad 4a^4b - 10a^3b^2 + 2a^2b^3 \\ \hline 6a^5 - 11a^4b - 7a^3b^2 + 2a^2b^3 \end{array}$$

In this operation, we have multiplied each term of the multiplicand by each term of the multiplier; and the number of partial products formed is, therefore, equal to the product of the number of terms in the multiplicand by the number of terms in the multiplier. But, in consequence of the reduction of similar terms, the number of terms in the final product is less than the number of the partial products formed, the latter being six, the former four only. In these partial products there are two which are unlike any others, and being, therefore, incapable of reduction with others, appear without change in the final product. These are the term affected with the highest exponent of the principal letter in the multiplier and multiplicand, and the term affected with the lowest exponent of this letter in each, viz., $6a^5$ and $2a^2b^3$.

This being premised, let it now be proposed to divide $6a^5 - 11a^4b - 7a^3b^2 + 2a^2b^3$ by $2a^3 - 5a^2b + ab^2$

The work may be disposed as follows, the divisor being placed at the right hand of the dividend, and the quotient written in a line beneath it:

$$\begin{array}{l|l}
6a^5 - 11a^4b - 7a^3b^2 + 2a^2b^3 & 2a^3 - 5a^2b + ab^2 \\
\underline{6a^5 - 15a^4b + 3a^3b^2} & \overline{3a^2 + 2ab} \\
\quad 4a^4b - 10a^3b^2 + 2a^2b^3 & \\
\quad \underline{4a^4b - 10a^3b^2 + 2a^2b^3} &
\end{array}$$

Regarding the dividend as the product of the divisor by the quotient, we observe that $6a^5$, the term affected with the highest exponent of the letter a in the dividend, must be the partial product arising from the multiplication of $2a^3$, the term affected by the highest exponent of a in the divisor, by the term affected with the highest exponent of a in the quotient. We shall, therefore, obtain a term of the quotient by dividing $6a^5$ by $2a^3$, which gives $3a^2$. We have, then, $3a^2$ for a term of the quotient. Multiplying the divisor by this, and subtracting the product from the dividend, we take out of the dividend all the partial products depending upon the divisor and first term of the quotient. The remainder, therefore, must contain all the partial products arising from the multiplication of the divisor by the remaining terms of the quotient, and these only. This remainder, viz., $4a^4b - 10a^3b^2 + 2a^2b^3$, may be regarded as a new dividend; and reasoning upon this as before, the term $4a^4b$, divided by $2a^3$, will give a new term of the quotient. This is $2ab$. Multiplying the divisor by this, and subtracting the product from the dividend, the dividend is exhausted, and we have for the final quotient $3a^2 + 2ab$. This, it is evident, is the true quotient, since, multiplied by the divisor, it reproduces the dividend.

The divisor and dividend being arranged with reference to some common letter, we have the following rule for the division of polynomials:

1°. *Divide the first term of the dividend by the first term of the divisor, and set the result, with its proper sign, as the first term of the quotient.*

2°. *Multiply the divisor by this first term of the quotient, and subtract the product from the dividend.*

3°. *Divide the first term of the remainder by the first term of the divisor, place the result in the quotient with its proper sign, multiply and subtract as before, and continue the process until the dividend is exhausted.*

EXAMPLES.

1. Divide $20a^5b - 19a^4b^2 - 49a^3b^3 + 75a^2b^4 - 27ab^5$
by $4a^2b - 7ab^2 + 3b^3$.
Ans. $5a^3 + 4a^2b - 9ab^2$.

2. Divide $x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$
by $x + y$. Ans. $x^3 + x^2y + xy^2 + y^3$.

3. Divide $6a^4 + 19a^3b + 22a^2b^2 + 11ab^3 + 2b^4$
by $2a^2 + 3ab + b^2$.
Ans. $3a^2 + 5ab + 2b^2$.

37. In the process of multiplication, some of the terms, by reduction, may entirely disappear in the final result. Thus, if $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ be multiplied by $x - y$, the product will be $x^5 - y^5$, in which a number of the terms arising in the multiplication do not appear, these terms having been cancelled by reduction.

Let it now be proposed to divide $x^5 - y^5$ by $x - y$.

$$
\begin{array}{l|l}
x^5 - y^5 & x - y \\
\cline{2-2}
\underline{x^5 - x^4y} & x^4 + x^3y + x^2y^2 + xy^3 + y^4 \\
\quad x^4y - y^5 & \\
\quad \underline{x^4y - x^3y^2} & \\
\qquad x^3y^2 - y^5 & \\
\qquad \underline{x^3y^2 - x^2y^3} & \\
\qquad\quad x^2y^3 - y^5 & \\
\qquad\quad \underline{x^2y^3 - xy^4} & \\
\qquad\qquad xy^4 - y^5 & \\
\qquad\qquad \underline{xy^4 - y^5} &
\end{array}
$$

Here the terms which reäppear in the process of division are those which are cancelled in the process of multiplication.

EXAMPLES.

1. Divide $a^3 + b^3$ by $a + b$. ANS. $a^2 - ab + b^2$.
2. Divide $a^4 - 2a^2x^2 + x^4$ by $a^2 + 2ax + x^2$.
ANS. $a^2 - 2ax + x^2$.
3. Divide $64x^6 - 16a^3x^3 + a^6$ by $4x^2 - 4ax + a^2$
ANS. $16x^4 + 16ax^3 + 12a^2x^2 + 4a^3x + a^4$.
4. Divide $1 - 18z^2 + 81z^4$ by $1 + 6z + 9z^2$.
ANS. $1 - 6z + 9z^2$.
5. Divide $a^5 - b^5$ by $a - b$.
ANS. $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.
6. Divide $81x^4 - 18x^2 + 1$ by $9x^2 - 6x + 1$.
ANS. $9x^2 + 6x + 1$.
7. Divide $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$ by $a^3 + 3a^2x + 3ax^2 + x^3$. ANS. $a^3 - 3a^2x + 3ax^2 - x^3$..

38. Let it be proposed next to divide a by $1 - x$. The dividend is not divisible by the divisor. We may however, attempt the division according to the rules which have been explained, and continue it at pleasure.

$$
\begin{array}{l|l}
a & 1 - x \\
\cline{2-2}
\underline{a - ax} & a + ax + ax^2 + ax^3 + \dfrac{ax^4}{1 - x} \\
\quad ax & \\
\quad \underline{ax - ax^2} & \\
\qquad ax^2 & \\
\qquad \underline{ax^2 - ax^3} & \\
\qquad\quad ax^3 & \\
\qquad\quad \underline{ax^3 - ax^4} & \\
\qquad\qquad ax^4 &
\end{array}
$$

The operation, it is evident, would never terminate. The quotient in this case is called an *infinite series*. From the terms already obtained, it is evident that each term, after the first, is formed by multiplying the preceding term by x. This is called the *law* of the series. By means of this law, we may continue the quotient at pleasure, without performing any more operations.

At whatever point we stop, in order to complete the quotient, the remainder should be written over the divisor in the form of a fraction, and annexed to the quotient.

MISCELLANEOUS EXAMPLES.

1. Divide 1 by $1 + a$.

   ANS. $1 - a + a^2 - a^3 + a^4 -$ &c.

2. Divide $x^3 - y^3$ by $x + y$.

   ANS. $x^2 - xy + y^2 - \frac{2y^3}{x+y}$.

3. Divide $x^5 + y^5$ by $x + y$.

   ANS. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

4. Divide $2a^4 - 13a^3b + 31a^2b^2 - 38ab^3 + 24b^4$ by $2a^2 - 3ab + 4b^2$.

   ANS. $a^2 - 5ab + 6b^2$.

5. Divide $81a^8 + 16b^{12} - 72a^4b^6$ by $9a^4 + 12a^2b^3 + 4b^6$.

   ANS. $9a^4 - 12a^2b^3 + 4b^6$.

6. Divide $a^5 + a^4x - a^3x^2 - 7a^2x^3 + 6x^5$ by $a^2 - x^2$.

   ANS. $a^3 + a^2x - 6x^3$.

7. Divide $5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$ by $5a^4 - 2a^3b + 4a^2b^2$.

   ANS. $a^3 - 4a^2b + 2b^3$.

SECTION VI. — ALGEBRAIC FRACTIONS.

39. A fraction in Algebra has the same signification as a fraction in Arithmetic. The denominator shows the number of parts into which the quantity taken as unity is divided, and the numerator shows the number of parts which are taken. Thus, in the fraction $\frac{a}{b}$, unity is divided into b parts, and a of these parts are taken.

An algebraic expression, partly entire and partly fractional, is called a *mixed* quantity.

The nature of the fractions being the same, the rules for the operations upon them will be the same. We shall merely subjoin these rules, with some examples under each.

1. TO REDUCE A MIXED QUANTITY TO A FRACTIONAL FORM.

RULE. *Multiply the integral part of the quantity by the denominator of the fraction, and add the numerator to the product. The sum will be the numerator of the required fraction.*

Reduce the following quantities to a fractional form:

1. $b + \frac{b}{a}$. ANS. $\frac{a\,b + b}{a}$.

2. $a^2 + \frac{b^2}{c}$. ANS. $\frac{a^2 c + b^2}{c}$.

3. $a - x + \frac{x^2}{a + x}$. ANS. $\frac{a^2}{a + x}$.

4. $b + \frac{a\,b}{a + b}$. ANS. $\frac{2\,a\,b + b^2}{a + b}$.

5. $3\,a\,b - \frac{a^2 + 3\,a^2 b^2}{a\,b}$. ANS. $-\frac{a}{b}$.

6. $a + x - \frac{a^2 + x^2}{a - x}$. Ans. $-\frac{2x^2}{a - x}$.

7. $a^2 - ax + x^2 - \frac{x^3}{a + x}$. Ans. $\frac{a^3}{a + x}$

8. $x + y - \frac{x^2 - y^2 + y}{x - y}$. Ans. $\frac{y}{y - x}$

9. $a^2 - ab + b^2 - \frac{2b^3}{a + b}$. Ans. $\frac{a^3 - b^3}{a + b}$

2. TO REDUCE FRACTIONAL EXPRESSIONS TO A MIXED QUANTITY.

RULE. *Divide the numerator by the denominator, and annex the fractional remainder, if any, to the quotient.*

Reduce the following fractional expressions to mixed quantities:

1. $\frac{4ab - b^2}{a}$. Ans. $4b - \frac{b^2}{a}$.

2. $\frac{a + b}{a - b}$. Ans. $1 + \frac{2b}{a - b}$.

3. $\frac{2x^2 + 2y^2}{x + y}$. Ans. $2x - 2y + \frac{4y^2}{x + y}$.

4. $\frac{27a^3 - 3b^2 - 4x + 9a^2}{9a^2}$.

Ans. $3a + 1 - \frac{3b^2 + 4x}{9a^2}$.

5. $\frac{6a^2 + 5ax - x^2}{3a^2 + 2ax}$. Ans. $2 + \frac{ax - x^2}{3a^2 + 2ax}$.

3. TO REDUCE FRACTIONS TO A COMMON DENOMINATOR.

RULE. *Multiply all the denominators together for a new denominator, and each numerator by all the denominators except its own, for a new numerator.*

Reduce the following fractions to a common denominator

1. $\frac{a}{b}$, and $\frac{c}{d}$. Ans. $\frac{a\,d}{b\,d}$, and $\frac{b\,c}{b\,d}$.

2. $\frac{x}{2}, \frac{x}{3}$, and $\frac{x}{5}$. Ans. $\frac{15\,x}{30}, \frac{10\,x}{30}$, and $\frac{6\,x}{30}$.

3. $\frac{x}{a+b}$, and $\frac{x}{a-b}$. Ans. $\frac{a\,x-b\,x}{a^2-b^2}$, and $\frac{a\,x+b\,x}{a^2-b^2}$.

4. $\frac{3}{4}, \frac{a^2}{3}, a+\frac{2\,x}{a}$. Ans. $\frac{9\,a}{12\,a}, \frac{4\,a^3}{12\,a}, \frac{12\,a^2+24\,x}{12\,a}$.

If the denominators of the fractions have common factors, a result may be obtained more simple than that derived from the rule. Thus, let it be required to reduce to a common denominator

$$\frac{x}{a-b}, \quad \frac{x}{a^2-b^2}, \quad \frac{x}{a+b}.$$

If we multiply the first by $a+b$ and the third by $a-b$, the fractions will be reduced to a common denominator, a^2-b^2. This is the same as dividing their least common multiple by each of the denominators, and then multiplying both terms of each of the fractions respectively by the quotient.

4. TO REDUCE FRACTIONS TO THEIR LOWEST TERMS.

Rule. *Divide by any quantity that will exactly divide both terms of the fraction.*

Reduce the following fractions to their lowest terms:

1. $\frac{a^2\,b}{a^5\,b^2}$. Ans. $\frac{1}{a^3\,b}$.

2. $\frac{a-x}{a^2-x^2}$. Ans. $\frac{1}{a+x}$.

3. $\frac{a\,x+x^2}{a\,b^2+b^2\,x}$. Ans. $\frac{x}{b^2}$.

4. $\frac{x^3 + y^3}{x^2 - y^2}$. Ans. $\frac{x^2 - xy + y^2}{x - y}$.

5. $\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$. Ans. $\frac{2}{3}$.

6. $\frac{x^3 - b^2 x}{x^2 + 2bx + b^2}$. Ans. $\frac{x^2 - bx}{x + b}$.

In general, to reduce fractions to their lowest terms, we divide each term by the greatest common divisor. The method above is sufficient for the purposes of this treatise.

5. MULTIPLICATION OF FRACTIONS.

RULE. *Multiply the numerators for a new numerator, and the denominators for a new denominator.*

In the examples under this and the following rules, the results given by the rule are reduced to their lowest terms.

Multiply together the following fractions:

1. $\frac{3a^2 b}{4cd}$, and $\frac{4b}{7a^2 d}$. Ans. $\frac{3b^2}{7cd^2}$.

2. $\frac{3a}{4bx}$, $\frac{6x^2}{5a^2}$, and $\frac{15ax}{3b^2}$. Ans. $\frac{9x^2}{2b^3}$.

3. $\frac{a^2 + b^2}{a^2 - b^2}$, and $\frac{a - b}{a + b}$. Ans. $\frac{a^2 + b^2}{a^2 + 2ab + b^2}$.

4. $\frac{2x}{5}$, $\frac{3ax}{4b}$, and $-\frac{8x}{a}$. Ans. $-\frac{12x^3}{5b}$.

5. $\frac{4x - 12}{5x}$, and $\frac{x + 3}{4}$. Ans. $\frac{x^2 - 9}{5x}$.

6. $3x$, $\frac{x + 2}{3b}$, and $\frac{x - 2}{b + c}$. Ans. $\frac{x^3 - 4x}{b^2 + bc}$.

7. $x - \frac{x - 4}{5}$, and $\frac{x}{4}$. Ans. $\frac{x^2 + x}{5}$.

8. $a + \frac{ax}{a - x}$, and $x - \frac{ax}{a + x}$. Ans. $\frac{a^2 x^2}{a^2 - x^2}$.

6. DIVISION OF FRACTIONS.

RULE. *Invert the divisor, and proceed as in multiplication.*

Divide the following fractions:

1. $\frac{a^2 b}{c^2 d}$ by $\frac{a b}{c d}$. Ans. $\frac{a}{c}$.

2. $\frac{4 x + 12}{7}$ by $\frac{3 x + 9}{14 x}$. Ans. $\frac{8 x}{3}$.

3. $\frac{a + b}{x - y}$ by $\frac{x + y}{a - b}$. Ans. $\frac{a^2 - b^2}{x^2 - y^2}$.

4. $\frac{9 y^2 - 3 y}{7}$ by $\frac{3 y^2}{7}$. Ans. $\frac{3 y - 1}{y}$.

5. $\frac{3 x + 5}{a + b}$ by $\frac{15 x + 25}{a^2 - b^2}$. Ans. $\frac{a - b}{5}$.

6. $\frac{3 x^2}{5 x - 10}$ by $\frac{2 x}{15 x - 30}$. Ans. $\frac{9 x}{2}$.

7. $\frac{x - y}{x + y}$ by $\frac{x^2 - y^2}{x^2 + 2 x y + y^2}$. Ans. 1.

8. $b + \frac{b^2 + b x}{b - x}$ by $b - \frac{b^2 - b x}{b + x}$. Ans. $\frac{b^2 + b x}{b x - x^2}$.

9. $x^2 + \frac{x^4}{a^2 - x^2}$ by $\frac{a x}{a - x} - x$. Ans. $\frac{a^2}{a + x}$.

7. ADDITION OF FRACTIONS.

RULE. *Reduce the fractions to a common denominator, then add the numerators, and place the sum over the common denominator.*

Add the following fractions:

1. $\frac{a}{b}$, and $\frac{c}{d}$. Ans. $\frac{a d + b c}{b d}$.

2. $\frac{3a}{5a}$, $\frac{x}{4b}$, and $\frac{2x}{c}$.

Ans. $\frac{12bca + 5acx + 40abx}{20abc}$

3. $3x$, $x + \frac{3a}{4}$, and $4x - \frac{6a}{5}$. Ans. $8x - \frac{9a}{20}$.

4. $\frac{a}{a+z}$, and $\frac{z}{a-z}$. Ans. $\frac{a^2+z^2}{a^2-z^2}$.

5. $\frac{a}{x}$, $-\frac{b}{x^2}$, and $\frac{1}{x^3}$. Ans. $\frac{ax^2 - bx + 1}{x^3}$.

6. $\frac{1+x}{1-x}$, and $\frac{1-x}{1+x}$. Ans. $\frac{2+2x^2}{1-x^2}$.

7. $\frac{1}{1+x}$, and $\frac{1}{1-x}$. Ans. $\frac{2}{1-x^2}$.

8. $\frac{x}{x-1}$, and $\frac{x}{x+2}$. Ans. $\frac{2x^2+x}{x^2+x-2}$.

9. $\frac{1+x^2}{1-x^2}$, and $\frac{1-x^2}{1+x^2}$. Ans. $\frac{2(1+x^4)}{1-x^4}$.

10. $\frac{a^2}{a^2-x^2}$, $\frac{1}{a-x}$, and $-\frac{a}{a+x}$.

Ans. $\frac{ax+a+x}{a^2-x^2}$.

8. SUBTRACTION OF FRACTIONS.

Rule. *Reduce the fractions to a common denominator; then place the difference of the numerators over the common denominator.*

1. From $\frac{3a}{b}$ take $\frac{2a}{b}$. Ans. $\frac{a}{b}$.

2. From $\frac{x}{y}$ take $\frac{z}{v}$. Ans. $\frac{vx - yz}{vy}$.

3. From $5x$ take $\frac{3-2x}{6}$. Ans. $\frac{32x-3}{6}$.

4. From $\frac{a+b}{a-b}$ take $\frac{a-b}{a+b}$. Ans. $\frac{4ab}{a^2-b^2}$.

5. From $\frac{7a-6}{3}$ take $\frac{9a+3}{5}$. Ans. $\frac{8a-39}{15}$.

6. From $\frac{1}{x+y}$ take $\frac{1}{x-y}$. Ans. $-\frac{2y}{x^2-y^2}$.

7. From $\frac{a^3+b^3}{a-b}$ take $\frac{a^3-b^3}{a+b}$. Ans. $\frac{2a^3b+2ab^3}{a^2-b^2}$.

8. From $\frac{2y^2-2y+1}{y^2-y}$ take $\frac{y}{y-1}$.

Ans. $\frac{y-1}{y}$.

9. From $5x+\frac{4x-6}{7}$ take $2x+\frac{7x-12}{13}$.

Ans. $3x+\frac{3x+6}{91}$.

10. From $6a+\frac{3x-2a}{a}$ take $2a+\frac{4a-3x}{x}$.

Ans. $4a+\frac{3x^2+ax-4a^2}{ax}$.

11. From $\frac{4}{1-x}+\frac{7}{1+x}$ take $\frac{1+2x}{1-x^2}$.

Ans. $\frac{10-5x}{1-x^2}$.

SECTION VII.—Equations of the First Degree

40. The rules obtained in the preceding equations are sufficient for the solution of all equations of the first degree. We proceed to some examples involving operations, chiefly fractional, a little more complicated than those which have thus far been required.

1. A student spends a seventh part of his income for books, and the rest in his ordinary expenses; could he, however, receive an addition of $100 yearly, then he might spend a fifth part of his annual income in books, and still have $40 more than before for his ordinary expenses. What was his income? ANS. $700.

2. I take a certain number, multiply it by $3\frac{3}{7}$, take 60 from the product, multiply the remainder again by $2\frac{1}{2}$, and subtract again 30, when nothing remains. What is the number? ANS. 21.

3. A person possesses a wagon with a mechanical contrivance, by which the difference of the number of revolutions of the wheels on a journey may be determined. It is known that each of the fore-wheels is $5\frac{1}{4}$ and each of the hind-wheels $7\frac{1}{8}$ feet in circumference. Now, when, in a journey, the fore-wheels have made 2000 revolutions more than the hind-wheels, how great was the distance travelled?

Let $x =$ the number of revolutions of the hind-wheels; then $x + 2000 =$ those of the fore, and we obtain $x = 5600$; whence the distance travelled will be 39,900 feet.

4. A boy had a certain number of pears and three times as many apples; after having sold 10 pears and bought 20 apples, he found that if he subtracted $\frac{3}{4}$ of his number of pears from his number of apples, the remainder would be the same as if he subtracted $\frac{1}{2}$ his number of apples from 5 times his number of pears. How many of each had he at first? ANS. 70 pears and 210 apples.

5. A gentleman deposited in Bank a certain sum of money. From this he drew a third part, and then deposited $50. A short time after, he drew from the sum thus augmented the fourth part, and again deposited $70. After this, he found that he had in Bank $120. What was the sum originally deposited?

Let $x =$ the original sum; then $\frac{2}{3}x + 50 = \frac{2x + 150}{3}$ $=$ what he had in Bank after drawing out the third part and depositing \$50. Proceeding in like manner, we obtain for the equation of the question

$$\frac{3(2x + 150)}{12} + 70 = 120;$$

whence $x = 25$. ANS. \$25.

6. A gentleman paid out $\frac{1}{4}$ of the money he had in his purse, and then received 3 shillings; again, he paid out $\frac{1}{3}$ of what he then had, and afterwards received 2 shillings; lastly, he paid out $\frac{1}{6}$ of what he then had, and found that he had 14 shillings left. How much had he at first? ANS. 24*s*.

7. A shepherd, driving a flock of sheep in time of war, meets with a company of soldiers, who plunder him of half of his flock and half a sheep over; and a second, third and fourth company treat him in the same manner, each taking half of the flock left by the last and half a sheep over, when but eight sheep remained. How many sheep had he at first?

ANS. 143.

8. A merchant increased his capital the first year by \$200 more than the half of it, and in the second year by \$300 more than one-half of what he had at the beginning of that year. He now found that his capital was three times as large as it was at first. What was it at first?

ANS. \$800.

9. A person has a certain sum of money placed to his credit. Upon this he draws for \$50 more than its half, upon the remainder he draws for \$30 more than its fifth part, and again upon this last remainder he draws for \$20 more than its fourth part. He afterwards settles his account, and finds that he has only \$10 left. What was the sum placed to his credit?

Let $x =$ the sum; then $\frac{x}{2} - 50 = \frac{x - 100}{2} =$ what he had left after the first draft.

And $\frac{x-100}{2} - \frac{x-100}{10} - 30 = 4x - \frac{700}{10} =$ what he had left after the second draft.

The equation of the question will then be

$$\frac{4x - 700}{10} - \frac{4x - 700}{40} - 20 = 10;$$

whence $x = 275$. Ans. $275.

10. A father, dying, leaves his estate to be divided among his three children in the following manner: the youngest child is to receive $200 and one-half of the remainder; the second is to receive one-fifth of the residue and $400; the eldest is to receive the remainder, which amounts to $520. What was the value of the estate?

Ans. $2500.

11. Two persons, A and B, found a purse containing dollars. A took from it $2 and the sixth of the remainder, after which B took from it $3 and the sixth of the remainder, when it was found that they had taken from it equal sums. How much money was there in the purse at first?

Ans. $20.

12. A merchant adds yearly to his capital one-third, but takes from it, at the end of each year, $1000 for his expenses. At the end of the third year, after deducting the last $1000, he finds himself in possession of twice the sum he had at first. How much did he possess originally?

Ans. $11100.

13. Out of a certain sum a man paid his creditors $96; half of the remainder he lent his friend; he then spent one fifth of what now remained, and, after all these deductions,

had one-tenth of his money left. How much had he at first? Ans. $128.

14. If 10 apples cost a penny and 25 pears cost two-pence, and I buy 100 apples and pears for ninepence half-penny, how many of each shall I have?

Ans. 75 apples and 25 pears.

15. A trader took from his capital, to meet his expenses, $50 a year for three years in succession; and in each of those years augmented that part of his stock which was not so expended by one-third thereof. At the end of the third year, his original capital was doubled. What was that capital? Ans. $740.

PROBLEMS AND EQUATIONS WITH TWO UNKNOWN QUANTITIES.

41. We have solved the preceding equations by the aid of one unknown quantity only. It is often convenient, and even necessary, to employ more than one unknown quantity.

1. The sum of two numbers is 70, and their difference is 10. What are the numbers?

This question naturally presents itself with two unknown quantities, viz., the two numbers required. Let us put x for the greater and y for the less.

Then by the first condition, $x + y = 70$;
and, by the second, $x - y = 10$.

If we now deduce the value of x from the second equation, we obtain $x = 10 + y$. And if for x in the first equation we put its value, $10 + y$, we obtain

$$10 + y + y = 70,$$

an equation which contains only one unknown quantity, and from which we obtain $y = 30$.

Putting next, either in the first or second equation, for y

its value, 30, we obtain $x = 40$. Thus the two numbers are 30 and 40.

2. Two purses together contain $250. If, now, $25 are taken out of the first and put into the second, they will then contain each the same sum. How many dollars does each contain? Ans. $150, and $100.

3. A boy bought a peach and 4 apples for 14 cents. He afterwards bought, at the same price, a peach and 7 apples for 23 cents. What was the price of each?

Ans. 2 and 3 cents, respectively

4. Says A to B, Give me $100, and I shall have as much as you. No, says B, give me, rather, $100, and then I shall have twice as much as you. How many dollars had each? Ans. A $500, B $700.

5. Find two numbers such that the sum of five times the first and six times the second shall be equal to 11, and the difference between three times the first and twice the second shall be equal to 5.

Putting x and y for the numbers, the equations will be

$$5x + 6y = 11$$
$$3x - 2y = 5.$$

From the last equation we obtain

$$x = \frac{5 + 2y}{3}.$$

Substituting this value of x in the first equation,

$$\frac{25 + 10y}{3} + 6y = 11.$$

From which we obtain $y = \frac{2}{7}$, and, by substitution, $x = 1\frac{6}{7}$.

6. A draper sold 2 yards of broadcloth and 3 yards of velvet for $35; and afterwards he sold 4 yards of broadcloth and 5 yards of velvet for $65. What was the price of each per yard? Ans. $10 and $5.

7. A gentleman purchased 7 pairs of shoes and 3 pairs of boots for \$29. He afterwards purchased, at the same rate, 5 pairs of shoes and 2 pairs of boots for \$20. What was the price of the shoes and boots a pair?

ANS. The shoes \$2, and the boots \$5.

8. A farmer sells to one man 3 cows and 5 oxen for \$235. He afterwards sells to another, at the same rate, 7 cows and 10 oxen for \$490. For what did he sell each?

ANS. Cows at \$20, and oxen at \$35.

9. After A had won 4 shillings of B, he had only half as many shillings as B had left. But if B had won 6 shillings of A, then B would have had three times as many as A would have had left. How many had each?

ANS. A 36*s*., B 84*s*

10. The mast of a ship consists of two parts: one-third of the lower part added to one-sixth of the upper part is equal to 28 feet; and 5 times the lower part diminished by 6 times the upper part is equal to 12 feet. What is the length of each part? ANS. The lower 60, and the upper 48 feet.

ELIMINATION.

42. In solving the preceding questions, our first object has been to obtain, from the two equations with two unknown quantities given by the question, a single equation containing one unknown quantity only. This is called *elimination.* The particular process already explained is called *elimination by substitution.* It consists in deducing the value of one of the unknown quantities from one of the equations and substituting this value in the other.

1. Let us take next the equations

$$4x - 7y = 34$$
$$8x + 3y = 102.$$

From the first equation we have

$$x = \frac{34 + 7y}{4};$$

from the second, $x = \dfrac{102 - 3y}{8}$.

But since things which are equal to the same are equal to each other, we have

$$\frac{34 + 7y}{4} = \frac{102 - 3y}{8}.$$

From which we obtain $y = 2$; whence $x = 12$.

The method of elimination here employed is called *elimination by comparison.* It consists in finding the value of one of the unknown quantities in each of the equations, and then putting these values equal to each other.

Ex. 2. Find, by the same method, the values of x and y in the equations

$$x + 15y = 53$$
$$y + 3x = 27.$$

Ans. $x = 8$, $y = 3$.

Ex. 3. Find the values of x and y in the equations

$$5x + 4y = 58$$
$$3x + 7y = 67.$$

Ans. $x = 6$, $y = 7$.

43. 1. Let it be proposed to eliminate one of the unknown quantities, y for example, from the following equations:

$$x + y = 220$$
$$4x - 3y = 180.$$

If we multiply the first equation by 3, the coefficients of y in the two equations will be equal, and the equations become

$$3x + 3y = 660$$
$$4x - 3y = 180.$$

Since the members of an equation are equal quantities, if we now add these equations, member to member, the results will be equal, and we shall have

$$7x = 840;$$

an equation containing x only, and from which we obtain $x = 120$.

2. Let it be required next to eliminate x from the equations

$$2x + 3y = 31$$
$$5x + 4y = 53.$$

If we now multiply the first equation by 5 and the second by 2, the coefficients of x will be equal in the two equations which become

$$10x + 15y = 155$$
$$10x + 8y = 106.$$

On the same principle that we before added, if we now subtract the second of these equations from the first, we obtain

$$7y = 49;$$

whence

$$y = 7.$$

3. Let it be required next to eliminate x from the equations

$$4x + 9y = 51$$
$$8x - 13y = 9.$$

Here 4, the coefficient of x in the first equation, is a factor of 8, the coefficient of x in the second, the other factor being 2; if, then, we multiply the first equation by 2, the coefficients of x in the two equations will be equal. Performing the operation, and subtracting the second equation from the first, we obtain

$$31y = 93;$$

whence

$$y = 3.$$

4. As a fourth example, let x be eliminated from the following equations:

$$6x + 8y = 36$$
$$15x - 7y = 9.$$

Here the coefficients of x have a common factor 3, the remaining factor in the first equation being 2, and in the second 5. If, then, we multiply the first equation by 5 and the second by 2, the coefficients of x in each will be equal, and we have

$$30x + 40y = 180$$
$$30x - 14y = 18.$$

From these last we obtain $y = 3$, $x = 2$.

The process of elimination here explained is called *elimination by addition and subtraction.* It consists in multiplying one or both of the equations by such number or numbers as will make the coefficients of one of the unknown quantities the same in both, and then adding or subtracting, according as the terms, the coefficients of which are made equal, have different or the same signs.

In the application of this method, if the proposed equations are not in the form of the preceding examples, they must be brought to this state. That is, the equations must be freed from denominators, the unknown quantities collected each into one term on one side of the sign of equality, and the known quantities collected in one term on the other.

44. Either of the preceding modes of elimination may be employed, at pleasure. In general, the latter will be found most convenient in practice. Whichever the mode employed, care should be taken to commence with the unknown quantity which is least involved in the equations.

1. Find the values of x and y in the equations

$$x + y = 9$$
$$3x + 5y = 35.$$

Ans. $x = 5$, $y = 4$.

2. Find the values of x and y in the equations

$$x + 15y = 53$$
$$3x + y = 27.$$

Ans. $x = 8$, $y = 3$.

3. Find the values of x and y in the equations

$$2x + 3y = 17$$
$$3x - 2y = 6.$$

ANS. $x = 4, y = 3.$

4. Find the values of x and y in the equations

$$\frac{x}{3} - \frac{y}{4} = 2$$
$$\frac{x}{4} + \frac{y}{2} = 7.$$

ANS. $x = 12, y = 8.$

5. Find the values of x and y in the equations

$$\frac{x}{2} + \frac{y}{3} = 7$$
$$\frac{x}{3} + \frac{y}{2} = 8.$$

ANS. $x = 6, y = 12.$

6. Find the values of x and y in the equations

$$\frac{2x - y}{2} + 14 = 18$$
$$\frac{2y + x}{3} + 16 = 19.$$

ANS. $x = 5, y = 2.$

7. Find the values of x and y in the equations

$$\frac{2x + 3y}{6} + \frac{x}{3} = 8$$
$$\frac{7y - 3x}{2} - y = 11.$$

ANS. $x = 6, y = 8.$

45. The following questions will serve as an additional exercise in the solution of questions by the aid of two unknown quantities.

1. A gentleman has two purses. If he puts $8 into the

first, then it is half as valuable as the other; but if he takes the \$8 out of the first and puts them into the second, then this is worth three times as much as the first. What is the value of each purse?

Ans. The first \$48, the second \$112.

2. Two numbers are given by the following properties. If the first be increased by 4, it is $3\frac{1}{4}$ times as great as the second; but if the second be increased by 8, then it is half as great as the first. What are the two numbers?

Ans. 48 and 16.

3. A owes \$1200, B \$2500; but neither has enough to pay his debts. Lend me, said A to B, the eighth part of your fortune, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you will lend me the ninth part of yours. What was the fortune of each?

Ans. A's \$900 and B's \$2400.

4. A person has two large pieces of iron, whose weight is required. It is known that $\frac{2}{5}$ of the first piece weigh 96 lbs. less than $\frac{3}{4}$ of the other piece, and that $\frac{5}{8}$ of the other piece weigh exactly as much as $\frac{4}{9}$ of the first. How much did each of these pieces weigh?

Ans. The first 720 lbs., the second 512 lbs.

5. Said a lad to his father, How old are we? Six years ago, answered the latter, I was one-third more than three times as old as you; but three years hence, I shall be obliged to multiply your age by $2\frac{1}{6}$, in order to obtain my own. What is the age of each?

Ans. The father 36, the son 15 years.

6. There is a fraction such, that if 1 be added to the numerator, its value becomes $\frac{1}{3}$, and if 1 be added to the denominator, its value becomes $\frac{1}{4}$. What fraction is it?

Let $x =$ the numerator, and $y =$ the denominator. The fraction will be $\frac{x}{y}$, and we shall have, for the equations of the question, $\frac{x+1}{y} = \frac{1}{3}$, &c.

7. It is required to find a fraction such, that if 3 be subtracted from the numerator and denominator, it is changed into $\frac{1}{4}$, and if 5 be added to the numerator and denominator, it becomes $\frac{1}{2}$. What fraction is it? ANS. $\frac{7}{15}$.

8. Two clerks, A and B, send ventures, by which A gained $20 and B lost $50, when the former had twice as much as the latter; but had B gained $20 and A lost $50, then B would have had 4 times as much as A. What sum was sent by each? ANS. A sent $80, B $100.

9. A and B engage in trade. A gains $1500 and B loses $500, when A's money is to B's as 3 to 2; but had A lost $500 and B gained $1000, then A's money would have been to B's as 5 to 9. What was the stock of each?

ANS. A's $3000, B's $3500.

10. A certain number, consisting of two places of figures, is equal to 7 times the sum of its digits; and if 18 be subtracted from it, the digits will be inverted. What is the number?

Let $x =$ the figure in the place of tens, y that in the place of units; then the number will be $10x + y$, and we shall have, for the equations of the question,

$$10x + y = 7(x + y)$$
$$10x + y - 18 = 10y + x.$$

ANS. 42.

11. A certain number consists of two figures, the sum of which is 8; and if 36 be added to the number, the digits will be inverted. What is the number? ANS. 26.

12. Bought linen at 60 cts. per yard, and muslin at 15

cents per yard, amounting in all to \$11.40. I afterwards sold $\frac{1}{3}$ of the linen and $\frac{1}{4}$ of the muslin for \$3.89, having cleared 29 cents by the bargain. How many yards of each did I purchase? ANS. 15 of linen, 16 of muslin.

13. Purchased 25 lbs. of sugar and 36 lbs. of coffee for \$8.04; but the price of each having fallen 1 cent per lb., I afterwards bought 2 lbs. more of the first and 3 lbs. more of the second for the same money. What was the price of each? ANS. Sugar 12 cts., coffee 14 cts.

14. Bought 10 cows and 15 sheep for \$215. I afterwards purchased 5 cows and 7 sheep for \$107.50, the cows costing \$1 a head more and the sheep 50 cents less than before. What were the prices of the first lot?

ANS. Cows \$17, sheep \$3.

15. What fraction is that, whose numerator being doubled, and the denominator increased by 10, the value becomes $\frac{1}{3}$; but the denominator being doubled, and the numerator increased by 5, the value becomes $\frac{1}{2}$? ANS. $\frac{3}{8}$.

16. Not long since, the bushel of rye was 42 cents and the bushel of wheat 90 cents cheaper than they are now; then the price of rye was $\frac{2}{3}$ that of the wheat; their present prices are as 7 to 12. What is the present price of each?

ANS. Rye \$1.26, wheat \$2.16.

17. A grain-dealer makes a mixture of barley and oats. If he mixes 5 bushels of barley with 4 of oats, the mixture is worth 60 cents a bushel; if he mixes 7 bushels of barley with 3 of oats, the mixture is worth 66$\frac{1}{2}$ cents a bushel. What is the cost per bushel of the barley and oats?

ANS. The barley 80 cents, the oats 35 cents.

18. A wine-merchant has two kinds of wine. If he mixes 3 gallons of the worst with 5 of the best, the mixture is worth \$1 per gallon; but if he mixes 3$\frac{1}{4}$ gallons of the

worst with $8\frac{3}{4}$ gallons of the best, then the mixture is worth \$1.03$\frac{1}{3}$ per gallon. What does each wine cost per gallon?

ANS. The best \$1.12, the worst 80 cents.

19. 34 lbs. of zinc lose 5 lbs. when immersed in water, and 17 lbs. of tin lose 2 lbs. in water. A composition of zinc and tin weighing 136 lbs. loses 19 lbs. in water. How much does this composition contain of each metal?

Let $x =$ the number of lbs. of zinc, $y =$ the number of lbs. of tin; then $x + y = 136$. And since 34 lbs. of zinc lose 5 lbs., 1 lb. will lose $\frac{5}{34}$, and x lbs. will lose $\frac{5}{34}\,x$ lbs. In like manner, y lbs. of tin will lose $\frac{2}{17}\,y$ lbs., and we have $\frac{5}{34}\,x + \frac{2}{17}\,y = 19$. ANS. 102 lbs. zinc, and 34 lbs. tin.

20. 77 lbs. of gold lose 4 lbs. in water, and 21 lbs. of silver lose 2 lbs. in water. Now, if a composition of gold and silver, weighing $24\frac{1}{2}$ lbs., lose $1\frac{1}{2}$ lbs. in water, how many lbs. does it contain of each metal?

ANS. $19\frac{1}{4}$ lbs. of gold, and $5\frac{1}{4}$ lbs. of silver.

21. Hiero, King of Syracuse, having ordered his jeweller to make him a crown of gold, of the weight of 20 lbs., suspecting that he had put some silver in it, directed Archimedes to examine it. When weighed in water, it was found to lose $1\frac{1}{4}$ lbs. Required the number of lbs. of silver it contained, supposing that 19.64 lbs. of gold lose 1 lb. in water, and 10.5 lbs. silver lose 1 lb. in water.

ANS. 5.22....

PROBLEMS AND EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

46. 1. A mercer has 3 pieces of silk. He sells 2 yards of the first, 3 of the second, and 4 of the third, for \$19; again he sells 3 yards of the first, 2 of the second, and 5 of the third, for \$21; and again 5 of the first, 4 of the second, and 3 of the third, for \$23. At what price per yard did he sell from each piece?

This question presents itself obviously with three unknown quantities. Let x represent the price per yard of the first, y of the second, and z of the third. Then, by the question,

$$2x + 3y + 4z = 19$$
$$3x + 2y + 5z = 21$$
$$5x + 4y + 3z = 23.$$

If we now multiply the first of these equations by 3 and the second by 2, and subtract the latter from the former, we obtain

$$5y + 2z = 15.$$

Again, multiplying the first by 5 and the third by 2, and subtracting, we obtain

$$7y + 14z = 49.$$

We now have two equations, with two unknown quantities only. Eliminating from these, as in the preceding examples, we obtain $y = 2$, $z = 2.50$; and substituting these values for y and z in the first equations, we obtain $x = 1.50$. The prices are, therefore, \$1.50, \$2, and \$2.50, respectively.

EXAMPLES.

1. Find the values of x, y and z, in the equations

$$x + y + z = 15$$
$$x + 2y + 3z = 23$$
$$x + 3y + 4z = 28.$$

Ans. $x = 10$, $y = 2$, $z = 3$.

2. Find the values of x, y and z, in the equations

$$x + y + z = 9$$
$$x + 3y - 3z = 7$$
$$x - 4y + 8z = 8.$$

Ans. $x = 4$, $y = 3$, $z = 2$.

3. Find the values of x, y and z, in the equations

$$2x + 3y + 4z = 28$$
$$4x + 5y + 6z = 48$$
$$5x - 3y + 4z = 31.$$

ANS. $x = 5$, $y = 2$, $z = 3$.

4. Find the values of x, y and z, in the equations

$$\frac{2x + 3y}{2} + 2z = 8$$
$$3x + 2y - 5z = 8$$
$$\frac{5x - 6y}{3} + z = 2.$$

ANS. $x = 3$, $y = 2$, $z = 1$.

5. Find the values of x, y and z, in the equations

$$2x + y - 7z = 0$$
$$4x - 6y + 2z = 0$$
$$2x + 3y - 9z = 4.$$

ANS. $x = 5$, $y = 4$, $z = 2$.

2. Any number of unknown quantities may be employed, as convenience may require, provided there are as many distinct conditions in the question as there are unknown quantities employed. From what has been done, the method of resolving the equations will be readily inferred. Thus, if there are four equations with four unknown quantities, we combine the equations, two by two, until one of the unknown quantities is eliminated from the whole; we then have three equations, with three unknown quantities. In the same manner as in the preceding examples, we next combine these, two by two, until one of the unknown quantities is eliminated, and so on. The process is altogether similar, for five or more equations, with the same number of unknown quantities.

Ex. Find the values of x, y, z and u, in the following equations:

$$x+y-z+u=8$$
$$x-y+z+u=6$$
$$x+y+z-u=2$$
$$x-y-z+u=4.$$

ANS. $x=3$, $y=2$, $z=1$, $u=4$.

47. We pass to the solution of some questions involving more than two unknown quantities.

1. A wine-merchant has three kinds of wine, which he sells to his customers at the same price. To A he sells a gallon of each, for $4.50; to B two gallons of the cheapest, three of the second, and one of the first quality, for $8.50; to C, one gallon of the cheapest, two of the second, and three of the first quality, for $10. What was the price of each per gallon? ANS. $1, $1.50, and $2, respectively.

2. A grocer has three kinds of tea. When he mixes together 2 lbs. of the first, 4 lbs. of the second, and 2 lbs. of the third kind, the mixture is worth 77½ cents a pound; when he mixes together 5 lbs. of the first, 2 lbs. of the second, and 3 lbs. of the third, the mixture is worth 83 cents a pound; and when he mixes 3 lbs. of the first, 2 lbs. of the second, and 5 lbs. of the third kind, the mixture is worth 75 cents a pound. What is the worth of a pound of each? ANS. $1.75 cts., and 60 cts., respectively.

3. A, B and C, purchase sugar, coffee and tea, at the same prices. A pays $4.20 for 7 lbs. of sugar, 5 lbs. of coffee, and 3 lbs. of tea; B pays $3.40 for 9 lbs. of sugar, 4 lbs. of coffee, and 2 lbs. of tea; C pays $3.25 for 5 lbs. of sugar, 2 lbs. of coffee, and 3 lbs. of tea. What is the price of each per pound?

ANS. Sugar 10 cts., coffee 25 cts., and tea 75 cts.

4. Find three numbers such that their sum shall be 60; one-half of the first, one-third of the second, and one-fifth of the third, shall be 19; and twice the first, with 3 times the difference between the second and third, shall be 50.

ANS. The numbers are 10, 30 and 20.

5. A and B possess together only $\frac{2}{3}$ of the property of C; B and C have together $5\frac{2}{3}$ times as much as A; if B were \$400 richer than he actually is, then he would have as much as A and C together. How much has each?

Putting x, y and z, for y, their fortunes respectively, from the first condition we have $x + y = \frac{2}{3} z$,

or $$x + y - \frac{2}{3} z = 0, \text{ \&c.}$$

ANS. A \$120, B \$200, and C \$480.

6. To find three numbers such that the first, with $\frac{1}{2}$ of the sum of the second and third, shall be 119; the second, with $\frac{1}{5}$ of the difference between the third and first, shall be 68, and $\frac{1}{2}$ of the sum of the three numbers shall be 94.

ANS. 50, 63 and 75.

7. A, B and C, together, possess \$1500. If B gives A \$200 of his money, then A will have \$280 more than B; but if B receives \$180 from C, then both will have the same sum. How much has each?

ANS. A \$300, B \$420, and C \$780.

8. Three persons have to pay a debt of \$1350. Neither of them can pay it alone; but when they unite, it can be done in either of the following ways: 1st, by B's putting $\frac{1}{9}$ of his property to all of A's; 2dly, by C's putting $\frac{3}{4}$ of his property to that of B's, or by A's adding $\frac{3}{5}$ of his property to that of C's. How much did each possess?

ANS. A \$1250, B \$900, C \$600.

9. Three persons, A, B and C, compare their fortunes. Says A to B, Give me \$700 of your money, and I shall have

twice as much as you retain; says B to C, Give me \$1400 and I shall have thrice as much as you have remaining; says C to A, Give me \$420, and then I shall have five times as much as you retain. How much has each?

Ans. A \$980, B \$1540, C \$2380.

QUESTIONS PRODUCING NEGATIVE RESULTS.

48. From the nature of negative quantities, we have seen that the addition of a negative quantity is the same as the subtraction of an equal positive quantity, and that the subtraction of a negative quantity is the same as the addition of an equal positive quantity. It follows from this, that addition in Algebra does not always, as in Arithmetic, imply augmentation, but, on the contrary, either augmentation or diminution, according as the quantities to be added are positive or negative. So, likewise, subtraction does not always imply diminution, but, on the contrary, either diminution or augmentation, according as the quantities to be subtracted are positive or negative.

Let now the following questions be proposed:

1. The length of a certain field is 12 rods and the breadth is 8 rods; how much must be added to the length, that the field may contain 40 square rods?

Let $x =$ the quantity to be added;

then $\qquad 96 + 8x = 40;$

whence $\qquad x = -7.$

The answer is -7 rods. This is a true answer to the question in the algebraic sense of addition. For, if we substitute in the equation for x its value, -7, we obtain $96 - 56 = 40$, a true equation. The enunciation, however, intends an *arithmetical* addition, and in this sense the question involves an absurdity; since it requires that something should be added to 96 to make 40, when 96 is already greater than 40. This absurdity leads to the negative result, and is indicated by it.

If now for x in the equation we write $-x$, the equation becomes $96 - x = 40$, which will be the equation of the question, modified thus:

The length of a certain field is 12 rods and the breadth is 8 rods; how much must be *subtracted* from the length, that the field may contain 40 square rods?

Resolving the equation of the question, thus modified, we obtain $x = 7$. Here the answer is positive, and satisfies the question in the exact or arithmetical sense of the enunciation.

2. The length of a certain field is 13 rods and its breadth 7 rods; how much must be subtracted from its length, that the field may contain 119 square rods?

Let $x =$ the quantity to be subtracted;

then $$91 - 7x = 119;$$

whence $$x = -4.$$

The answer is -4 rods. Putting -4 for x in the equation, it becomes $91 + 28 = 119$, a true equation. But, as an *arithmetical* subtraction is intended in the enunciation, the question involves an absurdity, since we are required to subtract something from 91 to make 119, when 91 is already less than 119. If, however, we put $-x$ for x in the equation, it becomes $91 + 7x = 119$, which will be the equation of the question, modified thus:

The length of a certain field is 13 rods and its breadth 7 rods; how much must be *added* to its length, that the field may contain 119 square rods?

Resolving the equation of the question, thus modified, we obtain $x = 4$, a positive answer, and which satisfies the question in the sense of its enunciation.

3. A laborer wrought for a person 12 days, and had his son with him 7 days, and received 46 shillings; he afterwards wrought 8 days, having his son with him 5 days, and received 30 shillings; how much did he earn per day himself, and how much did his son earn?

Let $x =$ the daily wages of the man, and y that of the son; then, by the question,

$$12x + 7y = 46$$
$$8x + 5y = 30.$$

Resolving these equations, we obtain $x = 5$, $y = -2$.

If we now substitute for x its value, 5, in these equations, they become

$$60 + 7y = 46$$
$$40 + 5y = 30;$$

equations which are evidently absurd, since it is required to add something to 60 to make 46, and to 40 to make 30. If, however, we put $-y$ for y in the equations, they become

$$12x - 7y = 46$$
$$8x - 5y = 30.$$

The negative value, therefore, obtained for y, shows that the allowance made for the son, instead of *augmenting* the pay of the laborer, is a charge placed to his account. The equations obtained by the substitution of $-y$ for y correspond to the question modified thus:

A laborer wrought for a person 12 days, and had his son with him 7 days, at certain *expense*, and received 46 shillings. He afterwards wrought 8 days, and had his son with him 5 days, at expense as before, and received 30 shillings. How much did the laborer earn per day, and how much was charged him per day on account of his son?

Resolving the equations corresponding to the question thus modified, we obtain $x = 5$, $y = 2$.

4. What fraction is that, from the numerator of which if

1 be subtracted, the value of the fraction will be $\frac{4}{7}$ but if 1 be subtracted from the denominator, the value will be $\frac{3}{8}$?

Putting x for the numerator and y for the denominator of the fraction, we have, by the question,

$$\frac{x-1}{y}=\frac{4}{7},\quad \frac{x}{y-1}=\frac{3}{8}.$$

From these we obtain $x=-3$, $y=-7$.

Here the values of x and y are both negative. If now we put in the equations, $-x$, $-y$, for x and y, respectively, they become

$$\frac{-x-1}{-y}=\frac{4}{7},\quad \frac{-x}{-y-1}=\frac{3}{8};$$

or, since we can change all the signs in the numerator and denominator of a fraction without altering its value,

$$\frac{x+1}{y}=\frac{4}{7},\quad \frac{x}{y+1}=\frac{3}{8};$$

equations which correspond to the question, modified thus:

What fraction is that, to the numerator of which if 1 be *added*, the value of the fraction will be $\frac{4}{7}$; but if 1 be *added* to the denominator, the value will be $\frac{3}{8}$?

From what has been done, we see that a negative result indicates some absurdity in the enunciation of the question, arithmetically considered, and at the same time shows how to remove the absurdity, by rendering certain quantities *additive* which in the question were regarded as subtractive, or by rendering certain quantities *subtractive* which in the question were regarded as additive.

We see, also, that in order to rectify the question, we change in the equations the signs of the quantities found to be negative, and then make the conditions of the question correspond to these equations.

5. A father is 55 years old, and his son 16. In how many years will the son be one-fourth as old as the father?

Resolving the question, we obtain — 3 for the answer. The question should have been, How many years *ago* was the son one-fourth as old as the father?

6. What number is that whose fifth part exceeds its fourth part by 15?

The negative answer shows that the question should be, What number is that whose fourth part exceeds its fifth part? &c.

7. Two persons, A and B, comparing their fortunes, find that A's is to B's as 2 to 3; they gain each $40, when A's money is to B's as 4 to 7. How much money had each?

The negative answer shows that the persons were comparing their *debts*, instead of their money in hand.

8. A father is 45 years old, and his son 15 years; when will the father be 4 times as old as his son?

The negative answer shows that the father *was* four times as old as his son some years ago.

9. A cistern has three cocks; the first will fill it in 5 hours, the second in 8, and by the third it will be emptied in 3 hours. In what time will it be filled, if all three run together?

The negative answer shows that the cistern should be supplied by the *discharging* cock, and emptied by the other two.

10. Let the learner now explain the negative answers in this and the following questions:

A has $150, B $120; they each receive a certain sum, when it is found that A's money is to B's in the proportion of 3 to 2. What did each receive?

11. Two gentlemen comparing their fortunes, says A to

B, If the amount of my money be divided by yours, the quotient will be $\frac{3}{5}$; but if we lose each $1000, the quotient will be $\frac{2}{3}$. What was the fortune of each?

12. A courier starts from a certain place, and travels at the rate of 43 miles a day; ten days after, he is pursued by another, starting from the same place, and travelling at the rate of 34 miles a day. In how many days will the second courier overtake the first?

13. Three persons talking of their gains, says A to B, Yours and mine together would make $145; says B to C, Yours and mine together would make $50; says C to A, Yours and mine together would make $55. What was the gain of each?

14. A mechanic worked for a gentleman 10 days, having with him an apprentice 7 days and his son 3 days, and received $28.50. At another time, he worked 6 days, having with him his apprentice 4 days and his son 5 days, and received $13.50. At a third time, he worked 8 days, having his apprentice with him 3 days and his son 4 days, and received $17. How much did he receive per day himself, and how much for his apprentice and son, severally?

SECTION VIII.—Solution of Questions in a General Manner.

49. From what has been done, it will be seen that there are two distinct parts in the solution of a question: 1°, a process of reasoning, by which it is ascertained what numerical operations are necessary to obtain the answer; 2°, the performing the operations thus determined.

In the preceding examples, we have performed each numerical operation as soon as determined; and thus, while we have obtained the answer sought, no trace of the reasoning has been left in the result. Let us now perform an example so as to separate these two processes, conducting first the process of reasoning to its conclusion, and then performing the operations determined by it.

Let us take, for this purpose, the following question:

To divide the number 73 into two such parts that the greater shall exceed the less by 9.

Putting x for the less part, we have

$$x + x + 9 = 73$$
$$2x + 9 = 73$$
$$2x = 73 - 9$$
$$x = \frac{73 - 9}{2}.$$

We have here conducted by itself, in the use of algebraic language, the reasoning required for the solution of the question. The expression at which we have arrived is not the answer, but the result of the reasoning pursued, and by which we are taught how to obtain the answer. Translating this result into common language, it stands thus:

The less part is equal to one-half of the remainder, after subtracting 9 from 73.

Subtracting 9 from 73, we obtain 64; one-half of this is 32, the less part sought.

Let next the following question be proposed:

Two men, A and B, are to share between them \$125, in such a manner that B shall have \$15 more than twice as much as A. What will be the share of A?

Let $x =$ A's share; then

$$x + 2x + 15 = 125.$$

But x being taken once and also twice, or $1 + 2$ times we have

$$(1+2)\,x+15=125$$
$$(1+2)\,x=125-15$$
$$x=\frac{125-15}{1+2}.$$

Here, also, we have simply the conclusion of the reasoning pursued, according to which, in order to obtain the answer, we are required to subtract 15 from 125, and to divide the remainder by the sum of 1 and 2.

Performing the operations, we obtain \$36⅔ for the share of A, as required.

EXAMPLES.

Putting x for the answer, what are the operations necessary to obtain the value of x, or the answer sought, in the following questions:

1. What number is that to the double of which if 18 be added, the sum will be 82?

Ans. $x=\dfrac{82-18}{2}.$

2. What number is that from the quadruple of which if 14 be subtracted, the remainder will be 48?

Ans. $x=\dfrac{48+14}{4}.$

3. What number is that to 5 times which if we add twice the remainder after subtracting 9 from it, the sum will be 80?

Ans. $x=\dfrac{80+2\times 9}{5+2}.$

4. It is required to find a number such, that if 10 be added to it, ¾ of the sum will be 25.

Ans. $x=\dfrac{4\times 25-3\times 10}{3}.$

5. It is required to find a number such, that if it be diminished by 7, and 12 be added to $\frac{2}{3}$ of the remainder, the sum will be 70.

$$\text{Ans. } x = \frac{70 \times 3 + 2 \times 7 - 3 \times 12}{2}.$$

50. If the preceding examples are examined with attention, it will be seen that the reasoning process by which the operations are determined does not at all depend upon the particular numbers given in the question, but will be precisely the same, whatever these numbers may be. Let it be required, for example, to divide the number 100 into two such parts that the greater shall exceed the less by 25. Putting x for the less part, we have

$$x + x + 25 = 100$$
$$2x + 25 = 100$$
$$2x = 100 - 25$$
$$x = \frac{100 - 25}{2}.$$

Comparing this process with that pursued in the question first proposed, it will be seen that it differs in no respect from it, except that the number 100 has been substituted for 73, and 25 for 9. By conducting the reasoning by itself, then, we determine, not merely how to get the answer to the question proposed, but to all questions which differ from it only in the particular numbers given.

In the present case, suppose the number required to be divided is 175, and the given excess 50. Instead of solving the question anew, we are sure that we shall obtain the less part required by first subtracting 50 from 175, and then dividing the remainder by 2. Performing these operations, we obtain $62\frac{1}{2}$ for the less part required.

51. The method here pursued is called solving the question in a *general manner*, and the result obtained is called a *general solution* of the question.

Let us take some additional examples.

1. A prize of 1000 dollars is to be divided between two persons, A and B, whose shares are to each other as 7 to 9. What is the share of each?

Let $x =$ A's share. Resolving the question in a general manner, we obtain

$$x = \frac{1000 \times 7}{7 + 9}.$$

Performing the operations, we obtain, for A's share in this particular case, \$437.50; subtracting this from \$1000, we obtain \$562.50, for B's share. But we may resolve the question directly with reference to B's share. Thus, let $y =$ B's share; then

$$y = \frac{1000 \times 9}{7 + 9},$$

from which we obtain \$562.50 for B's share, as before.

As another particular case, suppose the prize to be \$1500, and the proportions as 5 to 13, what are the shares?

Multiplying, according to the expression for x, 1500 by 5, and dividing the product by $5 + 13 = 18$, we obtain \$416$\frac{2}{3}$ for A's share. In like manner, multiplying, according to the expression for y, 1500 by 13, and dividing by 18, we obtain \$1083$\frac{1}{3}$ for B's share.

What will be the shares of A and B, respectively, in the following particular cases? 1. Suppose the prize to be \$1150, and the shares as 5 to 7. 2. Suppose the prize to be \$3200, and the shares as 11 to 15.

2. The sum of \$600 is to be divided between three persons, A, B, and C, in such a manner that B is to have twice and C three times as much as A. What is the share of each?

Let $x =$ A's share; then

$$x = \frac{600}{1+2+3} = \text{A's share,}$$

and $$2x = \frac{600 \times 2}{1+2+3} = \text{B's share,}$$

and $$3x = \frac{600 \times 3}{1+2+3} = \text{C's share.}$$

For the particular case in question, we shall have A's share = \$100, B's = \$200, and C's = \$300.

1. What will be the share of A, if the sum to be divided is \$850, and if B has 3 times and C has 5 times as much as A?

2. What will be the share of B, if B has 7 and C has 4 times as much as A, and the sum to be divided is \$954?

3. What will be the share of C, if B has 6 and C has 11 times as much as A, and the sum to be divided is \$1250?

3. Divide \$1800 among three men in such a manner that B will have \$3 as often as A has \$2, and C \$7 as often as A has \$5.

Putting $x =$ A's share, we obtain

$$x = \frac{1800 \times 5 \times 2}{5 \times 2 + 5 \times 3 + 7 \times 2}.$$

What will be the expression for B's share? What for C's?

What will be the share of A, if the sum to be divided is \$3750, and if B has \$4 as often as A has \$3, and C has \$11 as often as A has \$9? Ans. \$1054.68. . . .

52. In solving questions in a general manner, as explained above, care must be taken not to perform any of the operations as we proceed, since by so doing we should defeat the object proposed. Suppose that in the last example we had inadvertently performed the multiplication of 5 by 2, the result,

$$x = \frac{1800 \times 10}{10 + 5 \times 3 + 7 \times 2},$$

would still give the value of x for this particular question, but would not show how to find its value for any other similar question. We should lose, therefore, the general solution, which should always teach us how to obtain the answer by means of the numbers given in the proposed question.

The liability to this error may be avoided, and, at the same time, the solution of the question in a general manner may be facilitated, by the use of some additional signs.

Returning to the question first solved, if we state this question without reference to any particular numbers, or, in other words, if we state the question itself in a general manner, it will stand thus:

To divide a given number into two such parts that the greater shall exceed the less by a given excess.

By the same principle that we put x to represent one of the parts sought, we may employ any convenient sign to represent the given number, and also the given excess.

We have agreed to represent unknown quantities, or the quantities sought in a question, by some one or more of the last letters of the alphabet, as x, y, z, &c. We will now represent the known, or given quantities, in the question, by the first letters of the alphabet, as a, b, c, &c.

Thus, in the present question, let a represent the number given to be divided, and b represent the given excess. The question then may be stated more concisely, thus:

To divide a number a into two such parts that the greater shall exceed the less by b.

Putting, as before, $x =$ the less part, the equation of the question will be

$$x + x + b = a;$$

from which we obtain $x = \frac{a - b}{2}$.

This expression for x is called a *formula* for x. It snows how, in all cases, the value of x may be obtained. Translated into common language, it is called a *rule*. Thus, in the present example, the rule by which to find the less part will be as follows:

From the given number subtract the given excess; one-half the remainder will be the less part.

To obtain a formula for the greater part, we may solve the question directly with reference to the greater part. But, since the greater part is equal to the less + the given excess, we shall have

$$\frac{a - b}{2} + b = \frac{a + b}{2} = \text{the greater part.}$$

The formula for the greater part will be, then,

$$\frac{a + b}{2};$$

from which we derive the rule for the greater part, viz.:

To the given number add the given excess; one-half the sum will be the greater part.

Examples. Find, by the rules, the two parts in the following cases:

1. Number to be divided, 183; given excess, 43.
2. Number to be divided, 193; given excess, 17.
3. Number to be divided, 754; given excess, 72.

If the preceding solution be examined with attention, it will be seen that the formula, which expresses the result of the reasoning in algebraic language, is much more concise than the rule, which expresses the same result in common language. Thus Algebra enables us not merely to express, in a concise manner, the reasoning process necessary to the solution of a question, but also, with like conciseness, the result or conclusion to which this process leads.

53. The following questions evidently imply the division of a number into two parts, one of which shall exceed the other by a given excess. They may, therefore, be solved by the formulas above.

1. A gentleman paid $350 for a horse and chaise; the chaise cost $125 more than the horse. What was the price of each?

2. A and B have together $250; but B has $75 more than A. How much has each?

3. A certain school contains 150 pupils. There are 30 females more than males. How many are there of each?

4. A house is supplied with water from a fountain by two pipes, one of which furnishes 20 gallons an hour more than the other. They both together furnish 50 gallons an hour. How much is furnished by each?

From these examples it will be seen that questions may be virtually the same, and be comprised in the same solution, though differing in the language in which they are expressed.

54. We proceed to the solution of some additional questions.

1. Two travellers set out, at the same time, from two places, A and B, which are 50 miles apart, and travel the direct road toward each other, until they meet. The one from A travels at the rate of 5 miles an hour, the one from B at the rate of 3 miles an hour. In how many hours will they meet?

To resolve this question generally, we suppose that the two towns are a given number of miles apart, which we will represent by a; that the traveller from A goes a certain number of miles an hour, which we will represent by b, and the one from B a certain number of miles an hour, which we will represent by c. Let $x =$ the number of hours in which they will meet; then

$$b x + c x = a;$$

whence $$x = \frac{a}{b + c}.$$

Translating this formula, we have the following rule by which to find the number of hours in which the travellers will meet, viz.: *Divide the number of miles the towns are apart by the sum of the number of miles the travellers each travel an hour.*

By the rule we obtain $6\frac{1}{4}$ hours for the answer to the question proposed.

Ex. 1. Suppose the distance is 375 miles, and that the travellers proceed at the rate of 7 and 9 miles, respectively; when will they meet?

Ex. 2. Suppose the distance is 543 miles, and the rates are 11 and 13 miles, respectively; when will the travellers meet?

2. A traveller sets out from a certain place, and travels at the rate of 5 miles an hour; 8 hours later, another traveller sets out from the same place, and travels at the rate of 9 miles an hour. In how many hours will the second traveller overtake the first?

To solve the question generally, let a represent the number of hours the first traveller has the start of the second, and let b and c be the rates at which they travel, respectively. Putting x for the time required, we obtain

$$x = \frac{a b}{b - c}.$$

Translating the formula, we have the following rule, by which to find the time required, viz.: *Multiply the number of hours the first traveller precedes the other by his rate, and divide by the difference of the rates.*

By the rule, we obtain 10 hours for the answer to the particular question proposed.

Ex. Suppose the first traveller sets out 12 hours before the other, and goes at the rate of 11 miles an hour; when will the second overtake him, travelling at the rate of 13 miles an hour?

3. A man purchased an equal number of pounds of coffee, sugar and tea, for which he paid $2.59. For the coffee he paid 9, for the sugar 8, and for the tea 20 cents per pound; how many pounds of each did he purchase?

To solve the question generally, let d represent the number of cents paid for the whole, and let the price per pound of the coffee, sugar and tea, be represented by a, b, and c, respectively. Putting x for the number of pounds of each, we obtain

$$x = \frac{d}{a + b + c}.$$

Translating this formula, we have, for the rule, *Divide the price of the whole by the sum of the prices of a pound of each sort; the quotient will be the number of pounds of each sort.*

In the particular case stated, the man purchased 7 pounds of each.

Ex. 1. Suppose the price of the whole $7.05; the coffee 12, the sugar 10, and the tea 25 cents a pound; how many pounds were there of each?

Ex. 2. Suppose the price of the whole $6.30; the coffee 17, the sugar 13, and the tea 33 cents a pound; how many pounds were there of each?

4. 50 persons, male and female, dine at a tavern at a

joint expense of $17. Each man pays 40 cents, and each woman 30 cents. How many men and women were there?

Let a represent the whole number of persons, d the joint expense, b what the men, c what the women paid each. Putting x for the number of men, we obtain

$$x = \frac{d - ac}{b - c}.$$

RULE. To find the number of men, *From the joint expense subtract the product of the whole number of persons by what the women paid each, and divide the remainder by the difference between what the men and women each paid.*

What will be the separate formula by which to find the number of women, and what will be the rule derived from this formula?

5. Find a number such that the sum of the quotients of this number by the numbers m and n, respecitvely, may be a.

Putting x for the number, we obtain

$$x = \frac{mna}{m + n}.$$

RULE. *Multiply the sum of the quotients by the product of the divisors of the number, and divide the product which results by the sum of the divisors.*

Ex. 1. What is the number when the sum of the quotients is 30, and the divisors are 7 and 3, respectively?

Ex. 2. What is the number when the sum of the quotients is 39, and the divisors are 8 and 5, respectively?

6. A gentleman, wishing to distribute the money he had in his purse among some poor persons, found that if he gave them a pence apiece, he would not have enough by d pence; but if he gave them b pence apiece, he would have c pence left. How many persons were there?

Putting x for the number, we obtain

$$x = \frac{c + d}{a - b}.$$

What is the rule derived from this formula; and what will be the number of persons, if when he tried 7 pence he had not enough by 15 pence, and when he tried 4 pence he had 9 pence over?

7. The fore-wheel of a carriage is a feet and the hind-wheel b feet in circumference; what will be the distance passed over, when the fore-wheel has made n revolutions more than the hind-wheel?

Putting x for the number of revolutions of the hind-wheel, we obtain

$$x = \frac{a n}{b - a};$$

multiplying this by b, we obtain, for the distance sought,

$$\frac{a b n}{b - a}.$$

What is the rule from this formula? and what will be the distance, when the hind-wheel is 7 feet and the fore-wheel is 5 feet in circumference, and the fore-wheel has made 50 more revolutions than the hind-wheel?

8. Two men engage to perform a certain piece of work; the first can do it alone in a days, and the second in b days. How long would it take both, working together, to do it?

The first can do $\frac{1}{a}$th part of it in one day; the second $\frac{1}{b}$th part. Putting x for the number of days, we have

$$\frac{x}{a} + \frac{x}{b} = 1;$$

whence

$$x = \frac{a b}{a + b}.$$

RULE. *Divide the product of the numbers expressing the times in which each could perform the work alone by the sum of these numbers.*

Ex. 1. If A can perform a certain piece of work in 7 days, and B can perform the same work in 5 days, how long will it take both together to perform it?

Ex. 2. A cistern is supplied with two pipes; the first will fill it in nine days, the second in 11 days; how long will it take both together to fill it?

This last question can be solved by the same formula with the preceding. Why is it so?

9. Divide the number a into two such parts that the first shall be to the second as m to n.

ANS. $\frac{m\,a}{m+n}$, and $\frac{n\,a}{m+n}$.

What is the rule from these formulas, and what will the parts be, 1°, when the given number is 75, and the required parts are as 3 to 2? 2°, when the given number is 63, and the required parts are as 7 to 5?

10. The sum of two numbers is a, and one is m times as large as the other. What are the numbers?

ANS. $\frac{a}{m+1}$, and $\frac{m\,a}{m+1}$.

What is the rule from these formulas? and what are some particular cases of the problem, and the answers to them by the rule?

11. A father is a years old, his son b years old. In how many years will the father be n times as old as his son?

Let $x =$ the number of years; then $x = \frac{a - n\,b}{n - 1}$.

Ex. 1. The father is 70, the son 20 years old. In how many years will the father be three times as old as his son?

Ex. 2. The father is 45, the son 15 years old. In how many years will the father be four times as old as his son?

Explain the negative answer in this case, and prepare the formula for the question as corrected by it.

12. Divide the number a into two such parts that their product shall be to the square of the greater part as m to n.

Let x be the greater part; then the equation of the question will be $n(ax - x^2) = mx^2$, or, dividing both members by x, $n(a - x) = mx$.

Ans. The parts are $\frac{an}{m+n}$, and $\frac{am}{m+n}$.

Ex. 1. What are the parts when the number is 54, and the required proportion is as 2 to 7?

Ex. 2. What are the parts when the number is 24, and the required proportion as 3 to 5?

13. If a be added to the first of two numbers, it becomes m times as great as the second; but if b be added to the second of the numbers, it becomes n times as great as the first. What are the numbers?

Let x and y represent the numbers; then, by the question,

$$x + a = my, \text{ and } y + b = nx;$$

whence

$$x - my = -a \quad (1)$$

$$nx - y = b. \quad (2)$$

Multiplying the second equation by m, and subtracting the first from the second,

$$mnx - x = mb + a;$$

whence

$$x = \frac{mb + a}{mn - 1}, \quad y = \frac{na + b}{mn - 1}.$$

Ex. 1. What are the numbers when $a = 9, b = 7, m = 3, n = 2$?

Ex. 2. What are the numbers when $a = 25, b = 8, m = 7, n = 4$?

14. If a be subtracted from the first of two numbers, it is then one-mth part of the second; but if b be subtracted from the second of the numbers, it is then one-nth of the first. What are the numbers?

Ans. $\frac{mna + nb}{mn - 1}$, and $\frac{mnb + ma}{mn - 1}$.

Ex. 1. What are the numbers when $a = 10$, $b = 5$, $m = 3$, $n = 2$?

Ex. 2. What are the numbers when $a = 12$, $b = 6$, $m = 5$, $n = 3$?

55. Required a general statement and solution of the following questions:

1. A gentleman divided $540 among three poor persons; to the second he gave twice as much as to the first, and to the third three times as much as to the second. What did he give to each?

2. A prize of $1500 is to be shared among three persons in the following manner: B is to have $250 more than A, and C $145 more than B. What is the share of each?

3. A father, dying, left $25000 to be divided between his wife, son and daughter; the son has $1000 more than twice as much as the daughter, and the wife $500 more than three times as much as the son. What was the share of each?

4. A person borrowed as much money as he had in his purse, and then spent 16 shillings; again he borrowed as much as he had left in his purse, after which he spent 16 shillings; he borrowed and spent in the same manner a third and fourth time; after the fourth expenditure, he had nothing left. How much money had he at first?

5. There are two numbers such that if one-fifth of the second be added to the first, the sum will be 40; and if one-

sixth of the first be added to the second, the sum will be 55. Required the numbers.

6. What fraction is that to the numerator of which if 2 be added, the value of the fraction will become $\frac{7}{9}$; but if 2 be added to the denominator, the value of the fraction will be $\frac{5}{11}$?

56. We have now seen the aid derived from the algebraic language in the solution of particular questions, and also in the solution of questions in a general manner. We derive similar aid from this language in all investigations which relate to the nature and properties of numbers, or the quantities represented by them.

Let it be required, for example, to ascertain what effect will be produced upon the value of a fraction, if the same quantity be added to both the numerator and denominator.

To take a particular case, let us add 6 to the numerator and denominator of $\frac{7}{12}$; the two fractions will then be

$$\frac{7}{12}, \text{ and } \frac{13}{18}.$$

If we now reduce these fractions to a common denominator, they become $\frac{126}{216}$, and $\frac{156}{216}$.

The numerator of the second fraction is evidently greater than the numerator of the first, while the denominators are the same. The proposed fraction is, therefore, increased by the addition of 6 to its numerator and denominator. But it does not follow that what has been shown to be the effect in this particular case will be equally so in respect to all other fractions. In order to make the proposition general, we must employ, in the investigation, signs which will represent any fraction whatever, and any number whatever that may be added to its terms. Let a and b be the numerator and denominator, respectively, of the fraction, and m the quantity to be added; the two fractions will then be

$$\frac{a}{b}, \text{ and } \frac{a+m}{b+m}.$$

Reducing to a common denominator, they become

$$\frac{a\,b+a\,m}{b^2+b\,m}, \text{ and } \frac{a\,b+b\,m}{b^2+b\,m}.$$

Comparing the numerators of these fractions, the part $a\,b$ it is evident, is the same in both, while the part $b\,m$ in the second is greater than the part $a\,m$ in the first, since, from the nature of the fraction, b is greater than a.

We now see that what we found to be true in respect to the particular fraction $\frac{7}{12}$ is true in respect to any other fraction, and that, generally,

If the same quantity be added to both terms of a fraction, the value of the fraction will be increased.

It is assumed, in this investigation, that the fraction is proper, — that is, that the numerator is less than the denominator; otherwise, the result will be reversed.

Let the learner now show what will be the effect, if the same quantity is subtracted from both terms of a fraction.

2. Let it be proposed next to ascertain the manner in which the parts of a quantity, consisting of two parts, are employed in forming the square of this quantity.

For this purpose, let a represent one of the parts and b the other; then we have

$$(a+b)^2 = a^2 + 2\,a\,b + b^2.$$

From which we learn, that if a quantity consists of two parts, the square of this quantity will be equal *to the square of the first part, plus twice the first part by the second, plus the square of the second.*

3. Let it next be proposed to determine the relation of

the product of the sum and difference of two quantities to these quantities themselves. For this purpose, let a be one of the quantities and b the other; then

$$(a + b)(a - b) = a^2 - b^2.$$

We thus learn that *the product of the sum and difference of two quantities is equal to the difference of their squares.*

These propositions are demonstrated, in Geometry, in another form, where it is shown, 1°, That if a line is composed of two parts, the square described upon the whole line is equal to the sum of the squares described upon the parts, plus twice the rectangle contained by the parts. 2°, That the rectangle contained between the sum and difference of two lines is equal to the difference of their squares.

Further illustrations of this kind will occur as we proceed The learner will now, however, see distinctly the nature of Algebra. It is a language, the object of which is to express concisely the various processes of reasoning required in mathematical investigations, and the results or conclusions to which these investigations lead.

SECTION IX.—Rules of Arithmetic.

57. By solving a question in a general manner, we obtain, as we have seen, a *rule*, by which we are enabled, not merely to find the answer to the proposed question, but to all others which differ from it only in the particular numbers which are given.

Every formula furnishes, by translation into common language, a rule. We may obtain, therefore, as many distinct rules as there are different problems to be solved. The rule being once obtained, the mere arithmetician may easily solve any question comprehended under it. He has merely to follow the directions of the rule. To investigate the rule is the work of the algebraist. In his hands alone is the instrument by which it is discovered.

We proceed to some additional illustrations of the aid derived from Algebra in determining general rules, by investigating a few of the more important practical rules usually found in treatises on Arithmetic.

RATIO AND PROPORTION.

58. There are two ways in which magnitudes, or the numbers which represent them, may be compared.

1°. We may wish to ascertain how much the greater of the two exceeds the less. In this case, we subtract the less from the greater, and the result is called their *ratio by difference.* Comparing, for example, the numbers 8 and 5, their ratio by difference is $8 - 5$, or 3. In like manner, the ratio by difference of the quantities a and b will be $a - b$, or $b - a$, according as the one or the other is the greater.

2°. We may wish to ascertain how often one magnitude is contained in the other. In this case, we divide the one by the other, and the result is called their *ratio by quotient.* Take, for example, the numbers 18 and 6; comparing the 18 with the 6, the ratio will be $\frac{18}{6} = 3$; comparing 6 with 18, the ratio will be $\frac{6}{18} = \frac{1}{3}$. In like manner, the ratio by quotient of a to b is $\frac{a}{b}$, and of b to a, $\frac{b}{a}$.

The ratio by difference is sometimes called *arithmetical* ratio, and that by quotient *geometrical* ratio. To these latter designations, which are not significant, the former are to be preferred.

The two magnitudes or numbers which constitute a ratio are called the *terms* of the ratio. The term first written is called the *antecedent*, and the other the *consequent*.

59. When two ratios by difference are equal, the four magnitudes or numbers which constitute them are called an *equidifference*. They are so called because they are the expression of two equal differences. Let there be the four numbers 13, 7, 18, 12. The difference between 13 and 7 is 6, and the difference between 18 and 12 is also 6. These four numbers are said, therefore, to form an equidifference, which is written thus:

$$13 \,.\, 7 : 18 \,.\, 12,$$

and is read thus: 13 is to 7 as 18 is to 12. This equidifference may also be written thus:

$$13 - 7 = 18 - 12.$$

In like manner, if the four quantities a, b, c, d are such that the difference between the first and second is equal to the difference between the third and fourth, these quantities form an equidifference, which is written thus:

$$a \,.\, b : c \,.\, d;$$

or thus: $$a - b = c - d.$$

When two ratios by quotient are equal, the four quantities or numbers which constitute them may, from this circumstance, be called an *equi-quotient*. They are usually, however, called a *proportion*. Let there be, for example, the four numbers 18, 6, 15, 5. The ratio of 18 to 6, or 18 divided by 6, is 3; so also, the ratio of 15 to 5 is 3. These numbers, therefore, form a proportion, which is written thus:

$$18 : 6 :: 15 : 5;$$

or thus:

$$\frac{18}{6} = \frac{15}{5};$$

and is read, 18 is to 6 as 15 is to 5.

In like manner, if the four quantities a, b, c, d are such that the ratio between the first and second is the same as the ratio between the third and fourth, these quantities form a proportion, and we have

$$a : b :: c : d,$$

or

$$\frac{a}{b} = \frac{c}{d}.$$

The first and last terms of an equidifference, or of a proportion, are called the *extremes*, and the second and third the *means* of the equidifference or the proportion.

PROPERTIES OF EQUIDIFFERENCES.

60. Let there be the equidifference $a \,.\, b : c \,.\, d$, or otherwise, $a - b = c - d$.

If we now transpose the terms b and d, we shall have the equation $a + d = c + b$.

From which we learn that in an equidifference *the sum of the extremes is equal to the sum of the means.* This is the fundamental property of equidifferences.

Conversely, if four quantities, a, b, c, d, are such that the sum of any two of them is equal to the sum of the other two, these quantities will form an equidifference. Thus let $a + d = c + b$. Transposing the d and b, the equation becomes $a - b = c - d$,

or $a \,.\, b : c \,.\, d$.

From the equidifference $a - b = c - d$, we derive

$$a = b + c - d$$

$$b = a + d - c.$$

If, then, three terms of an equidifference are given, we may find the fourth, if it be one of the extremes, by the following rule: *From the sum of the means subtract the given extreme; the remainder will be the required extreme.*

If the required part be one of the means, it will be found by the following rule: *From the sum of the extremes subtract the given mean; the remainder will be the required mean.*

It is sometimes the case that the two mean terms are the same; thus $a \,.\, b : b \,.\, c$. This is called a *continued* equidifference. In this case, we have

$$a + c = b + b;$$

whence

$$b = \frac{a + c}{2}.$$

Here b is called the *mean* between the quantities a and c. Thus, to find the mean of two quantities, *we take one-half the sum of these quantities.*

Ex. 1. A sum of money is divided among four persons in such a manner that their shares form an equidifference. The share of A is $12, that of B $15, that of C $17. What is the share of D?

Ex. 2. The first, second and fourth terms of an equidifference are 23, 28 and 37, respectively. What is the third term?

Ex. 3. What is the mean between 125 and 52?

PROPERTIES OF PROPORTIONS.

61. Resuming the general proportion $a : b :: c : d$, or

$$\frac{a}{b} = \frac{c}{d},$$

we obtain

$$a\,d = b\,c.$$

From this we learn that in a proportion *the product of the extremes is equal to the product of the means.*

This is the fundamental property of proportions. Any change may be made in the order in which the terms of a proportion are written, consistent with this fundamental property. Let there be, for example, the proportion

$$36 : 12 :: 75 : 25.$$

From this we may deduce the following proportions:

1°, by changing the extremes, $25 : 12 :: 75 : 36$.

2°, by changing the means, $36 : 75 :: 12 : 25$.

3°, by putting the means in the place of the extremes, $12 : 36 :: 25 : 75$.

In each of these, the ratio, it is evident, is different. The proportion is, however, preserved, since in all the changes the product of the extremes is equal to the product of the means.

Conversely, let there be four quantities, a, b, c, d, such that we have $a\,d = b\,c$;
dividing both sides by $b\,d$, we have

$$\frac{a}{b} = \frac{c}{d};$$

whence $a : b :: c : d$.

If, then, four quantities are such that the product of any two of them is equal to the product of the other two, these numbers will form a proportion, of which one of the products is the product of the extremes, and the other that of the means.

It may sometimes occur that the two mean terms in a proportion are equal, as in the proportion

$$a : b :: b : c.$$

This is called a *continued* proportion, and b is called a *mean proportional* between a and c.

From the continued proportion $a : b :: b : d$, we obtain

$$b^2 = a\,d.$$

Whence, in a continued proportion, *the square of the mean is equal to the product of the extremes.* To find, therefore, a mean proportional between two quantities, *we take the square root of the product of these quantities.*

62. Resuming again the general proportion,

$$a : b :: c : d, \text{ or } \frac{a}{b} = \frac{c}{d},$$

and adding unity to each side of this last, we obtain

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or, reducing to an improper fraction.

$$\frac{a+b}{b} = \frac{c+d}{d},$$

which gives $\quad a + b : b :: c + d : d.$

Comparing this with the original proportion, $a : b :: c : d$, we derive the following principle: In every proportion, *the sum of the first and second terms is to the second as the sum of the third and fourth terms is to the fourth.*

By a process altogether similar, it may also be shown that in every proportion *the difference between the first and second terms is to the second as the difference between the third and fourth is to the fourth.*

The proportion $a : b :: c : d$ returns to $a : c :: b : d;$ whence, from the first of the preceding properties,

$$a + c : c :: b + d : d.$$

Comparing this with the proposed, we derive the following principle: In every proportion, *the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent.*

In like manner it may be shown *that the difference of the antecedents is to the difference of the consequents as any one antecedent is to its consequent.*

Let there be next a series of quantities, a, b, c, d, e, f, g h ... forming, two and two, a series of equal ratios, so that we have

$$a : b :: c : d :: e : f :: g : h \ldots$$

The two first ratios, viz., $a : b :: c : d$, give, from the principle last demonstrated,

$$a + c : b + d :: c : d;$$

but $$c : d :: e : f;$$

whence $$a + c : b + d :: e : f;$$

applying to this last the same principle, we have

$$a + c + e : b + d + f :: e : f;$$

and thus in order, whatever the number of equal ratios. Therefore, *in a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent.*

Let us take next the two proportions

$$a : b :: c : d, \text{ and } e : f :: g : h;$$

or $$\frac{a}{b} = \frac{c}{d}, \text{ and } \frac{e}{f} = \frac{g}{h}.$$

If we now multiply these equations member by member, we obtain

$$\frac{ae}{bf} = \frac{cg}{dh};$$

whence $$ae : bf :: cg : dh.$$

Comparing his with the proposed proportions, the result is the same as if we had multiplied these proportions term by term. This is called multiplying the proportions *in order*. Hence, *if two proportions are multiplied in order, the results will be proportional.*

It readily follows, from this principle, that if the terms of a proportion are raised each to the same power, the results will be proportional. Conversely, if the same roots are taken of each of the terms of a proportion, the results will be in proportion.

RULE OF THREE.

63. This name is given to a rule in Arithmetic, by which, when three numbers are given, a fourth is found from them.

Thus, let there be the following question:

1. If 5 yards of cloth cost \$15, what will 7 yards of the same cloth cost?

Here there are three numbers given, viz., 5, 15 and 7, and it is required to obtain by means of these a fourth, viz., the price of the 7 yards. By analysis of the question, it is evident that if 5 yards cost \$15, then 2, 3, 4 ... times as as many yards will cost 2, 3, 4 ... times as much. There is, therefore, manifestly a proportion between the numbers which denote the yards, on the one part, and the numbers which denote the prices, on the other. We have thus given three terms of a proportion, by which to find the fourth. Let x denote the fourth term sought. The proportion will stand thus:

$$\overset{\text{yds.}}{5} : \overset{\text{yds.}}{7} :: \overset{\$}{15} : \overset{\$}{x},$$

and is read, 5 yards are to 7 yards as 15 dollars to x dollars. Deducing the value of x, we have

$$x = \frac{15 \times 7}{5} = 21.$$

Let a, b, c, represent, generally, the three given terms, and x the required term of the proportion; then

$$a : b :: c : x;$$

whence $$x = \frac{bc}{a}.$$

The three given numbers in the question being properly arranged, they will form the first three terms of a proportion; whence, to find the fourth term, or answer, *we multiply together the second and third terms, and divide the product by the first.*

Ex. 2. If 25 lbs. of sugar cost $6.50, what will 384 lbs. cost?

Here, also, there is a necessary proportion between the number of pounds, on the one hand, and the prices, on the other; and we shall have, therefore,

lbs. lbs. $
25 : 384 : : 6.50 : to the answer.

Applying the rule, we obtain for the answer, $99.84.

Ex. 3. If 45 men build 280 feet of wall in a day, how many feet will 75 men build in the same time?

From the question we have, evidently, the following proportion:

men. men. feet.
45 : 75 : : 280 : to the answer;

whence, by the rule, we obtain $466\frac{2}{3}$ feet for the answer.

Before applying the rule, the proportions may be simplified by striking out, where it can be done, common factors from the terms.

Thus, in the second question, the two antecedents have a common factor, 25; and in the third question, the first antecedent and its consequent have a common factor, 15, which may be struck out; by which, in each case, as it is easy to see, the final calculations will be rendered more simple.

In general, questions depending upon the Rule of Three contain in their enunciation four numbers, two of which are of a certain kind, and two others of a different kind, of which the required quantity or answer is one.

Thus, in the first question above, two of the numbers express yards, while the other two express the prices of these yards. In the last question, two of the numbers denote men, while the other two denote the amount of labor which the two sets of men are able to perform, respectively. In these two cases, it is evident that, as the number of yards is increased, the prices are increased, and at the same rate; and as the number of men is increased, the amount of labor is increased, and in the same proportion. The proportion between the four terms is, therefore, said to be *direct.*

In general, a proportion is said to be direct when the first two terms, being of the same kind, and the last two, being also of the same kind, increase or diminish at the same rate.

Ex. 4. If 135 men can perform a certain piece of work in 20 days, how long will it take 540 men to do the same work?

In this case, it is evident that the greater the number of men, the less will be the time required, and in the same proportion. Putting x for the answer, the proportion will stand thus:

$$\overset{\text{men.}}{135} : \overset{\text{men.}}{540} :: \overset{\text{days.}}{x} : \overset{\text{days.}}{20}.$$

Here the terms of the last ratio are inverted; whence the proportion is said to be *inverse.* In general, a proportion is said to be inverse, when, as the two terms of the first kind increase, the two of the second decrease, and the converse.

Inverting the order of the terms, so that x may become the fourth term, we have

$$540 : 135 :: 20 : x.$$

Applying the rule, we obtain $x = 5$. Whence the answer will be 5 days.

We have, therefore, the following rule for the statement and solution of all questions of the kind we are here considering, viz.:

1°. *Place the number which is of the same kind with the answer sought as the third term of a proportion, and then consider, from the nature of the question, whether the answer should be greater or less than this term.*

2°. *If the answer should be greater, write the less of the two remaining numbers as the first term, and the other as the second term, of the proportion; but if the answer is less than this third term, then write the greater of the two remaining numbers as the first term, and the other as the second term, of the proportion.*

3°. *Multiply the second and third terms together, and divide the product by the first term; the quotient will be the answer.*

Such is the *Rule of Three*, as it is usually given in Arithmetic. We subjoin a few additional examples.

Ex. 1. If 750 men require 22500 rations of bread for a month, how many rations will a garrison of 1200 men require? ANS. 36000.

Ex. 2. If 225 bushels of corn cost \$125, what will 375 bushels cost, at the same rate? ANS. $\$208\frac{1}{3}$.

Ex. 3. A garrison of 800 men is provisioned for 90 days. How many days will the provision last, if it receive a reinforcement of 300 men? ANS. $65\frac{5}{11}$ days.

Ex. 4. If 6 men can dig a certain ditch in 40 days, how many days would 30 men be employed in digging it?

ANS. 8 days.

Ex. 5. The weights of a lever have the same ratio as the lengths of the opposite arms. The ratio of the weights is 5, that is, they are to each other as 1 to 5, and the longer arm is 10 inches. What is the length of the shorter arm?

ANS. 2 inches.

Ex. 6. The weights of a lever are 6 and 8 pounds, and the length of the shorter arm is 18 inches. What is the length of the longer arm? ANS. 24 inches.

Ex. 7. The force of gravitation is inversely as the square of the distance. At the distance 1 from the centre of the earth, this force is expressed by the number 32.16. By what number is it expressed at the distance 60?

ANS. 0.0089.

2. Questions sometimes occur involving more than one proportion, as the following:

If 16 men build 18 feet of wall in 12 days, how many men, working at the same rate, must be employed to build 72 feet in 8 days?

To simplify the question, let it be supposed that the number of men employed is the same in each case; then the question will be,

If a certain number of men build 18 feet of wall in 12 days, how long will it take them to build 72 feet? which gives, putting x for the number of days,

$$\text{(1)} \quad \overset{\text{feet.}}{18} : \overset{\text{feet.}}{72} :: \overset{\text{days.}}{12} : \overset{\text{days.}}{x}.$$

Regarding x as determined, the question will then be,

If 16 men can build 72 feet of wall in x days, how many men will it take to build the same number of feet in 8 days? which, putting y for the number of men, gives

$$\text{(2)} \quad \overset{\text{days.}}{8} : \overset{\text{days.}}{x} :: \overset{\text{men.}}{16} : \overset{\text{men.}}{y}.$$

Multiplying next these two proportions in order, we have

$$\text{(3)} \quad 18 \times 8 : 72 \times x :: 12 \times 16 : xy;$$

or, striking out the common factor, x, from the two consequents,

$$\text{(4)} \quad 18 \times 8 : 72 :: 12 \times 16 : y;$$

whence $y = \dfrac{72 \times 12 \times 16}{18 \times 8} = 4 \times 12 \times 2 = 96.$

Introducing in the proportions the simplifications already explained, the work will stand thus:

$$\begin{array}{ll} (1) & 1 \,.\, 4 :: 12 : x \\ (2) & 1 \,.\, x :: \ 2 : y \\ (3) & 1 : 4 \times x :: 12 \times 2 : x y \\ (4) & 1 : 4 :: 12 \times 2 : y. \end{array}$$

$$y = 4 \times 12 \times 2 = 96.$$

We may easily deduce a rule for questions of this kind. such as is usually given in Arithmetic under what is termed *Compound Proportion*. The rule, however, is unnecessary, as all questions of the kind can be solved by means of two or more simple proportions.

In the present question, the value of x obtained from the first proportion is 48, which, substituted in the second, gives

$$\overset{\text{days.}}{8} : \overset{\text{days.}}{48} :: \overset{\text{men.}}{16} : \overset{\text{men.}}{y},$$

or $$1 : \ 6 :: 16 : y;$$

whence $$y = 96.$$

The question is solved, therefore, by means of two simple proportions, in conformity to the Rule of Three already obtained.

The following examples will serve as an exercise for the learner, in questions of this kind.

Ex. 1. If 20 men weave 84 yards of cloth in 6 days, how many days will 12 men take to weave 100 yards?

Ans. $11\frac{19}{21}$ days.

Ex. 2. If 25 persons consume 300 bushels of corn in one year, how much will 139 persons consume in 7 years, at the same rate? Ans. 11676 bushels.

Ex. 3. If a man travels 217 miles in 7 days, travelling 6 hours a day, how far would he travel in 9 days, if he travelled 11 hours a day? Ans. $511\frac{1}{2}$ miles.

Ex. 4. If 10 men dig 8 acres in 6 days, working 8 hours a day, how many men will be able to dig 7 acres in 3 days, working 10 hours a day? Ans. 14 men.

BARTER.

64. How much sugar, at 10 cents a pound, must be given for 58 gallons of wine, at 75 cents a gallon?

Let $x =$ the quantity; then, by the question,

$$10\,x = 58 \times 75;$$

whence $x = 435.$ Ans. 435 lbs.

The question may be stated generally thus:

How much of one commodity, at a given price, will be required in exchange for another commodity, at another given price?

Let a be the quantity of the given commodity, and b its price; x the required amount of the commodity to be given in exchange, and c its price; then

$$c\,x = a\,b;$$

whence $$x = \frac{a\,b}{c}.$$

From which we have the following rule for Barter: *Multiply the given quantity by its price, and divide the product by the price of the quantity required.*

Ex. 1. A farmer has corn at 5*s.* a bushel; how much must he give of this for 25 bushels of wheat at 9*s.* a bushel? Ans. 45 bushels.

Ex. 2. A dealer in fruit wishes to exchange apples worth \$2.25 a barrel for pears worth \$7.50 a barrel. How many bushels of apples must he give for 25 of pears? Ans. $83\frac{1}{3}$ bushels.

Ex. 3. How many pounds of lead, at 9 cents per pound must be given for 783 lbs. of iron, at 6 cents per pound?

Ans. 522 lbs.

Ex. 4. A has broadcloth at 16*s*. 6*d*. per yard. B has linen at 1*s*. 4*d*. per yard. How many yards of broadcloth must be given in exchange for 660 yards of linen?

Ans. $53\frac{1}{3}$ yds.

FELLOWSHIP.

65. The rule of Fellowship teaches how to divide among several persons, connected together in business, the gain or loss which results from their connection.

Three men, A, B and C, purchase in company a certain quantity of merchandise, amounting to $250, of which A pays $50, B $75, and C $125. They gain by the purchase $120. What is each man's share of the gain?

Let $x =$ A's share; then, since each man's share of the gain should be in proportion to the amount contributed by him towards the purchase, we shall have

$$\frac{75x}{50} = \text{B's share, and } \frac{125x}{50} = \text{C's share;}$$

whence $$x + \frac{75x}{50} + \frac{125x}{50} = 120.$$

Resolving the equation, we obtain $x = \$24 =$ A's share; whence B's share will be $36, and C's $60.

The question may be stated in a general manner thus:

Three men, A, B and C, trade together, and furnish money in proportion to the numbers m, n and p, respectively. They gain a certain sum, a. What is each man's share of the gain?

Since each man's share of the gain should be in proportion to the sum contributed by him to the common stock, we shall have, putting x, y and z, for the shares of each respectively, the following equal ratios:

$$x : m :: y : n :: z : p;$$

whence, taking the sum of the antecedents and that of the consequents,

$$x + y + z : m + n + p :: x : m$$
$$:: y : n$$
$$:: z : p;$$

but $\quad x + y + z = a;$

whence, by substitution, $a : m + n + p :: x : m;$

wherefore $\quad (m + n + p)\, x = m\, a,$

and $\quad x = \dfrac{m\, a}{m + n + p} =$ A's share.

Likewise $\quad y = \dfrac{n\, a}{m + n + p} =$ B's share,

and $\quad z = \dfrac{p\, a}{m + n + p} =$ C's share.

We have, therefore, the following rule, by which to find each man's share of the gain. *Multiply the whole gain by each man's proportion of the stock, and divide by the sum of the proportions.*

The process by which the rule is obtained, it is evident, will be the same, whatever the number of persons engaged in company. The rule is therefore general.

The following examples will serve as an exercise for the learner:

Ex. 1. Four persons, A, B, C, D, trade in company, and make a clear gain of \$1250. A put in \$750, B \$1450, C \$1575, and D \$2100. What is each man's share of the gain?

Ans. A's \$159.57, B's \$308.51, C's \$335.10, D's \$446.80.

Ex. 2. A, B and C, trade in company. A's stock is \$240, B's $360, and C's $600. They gain $325. What is each man's share of the gain?

ANS. A's $65, B's $97.50, and C's $162.50.

Ex. 3. A man, dying, leaves property to the amount of $3000. A has a note of $600 against the estate, B has a note of $1800, and C a note of $1600. How much must each lose? ANS. A $150, B $450, and C $400.

66. In the preceding examples, each man's stock has been employed for the same time. This is called *Single Fellowship*. But the times may also be different for which the stock of each man is employed in trade. This is called *Double Fellowship*.

Let us now make a rule for Double Fellowship. Before proceeding to this, we remark that when we have occasion to represent several things of a kind, it is convenient to represent each by the same letter, and to distinguish between them by means of accents applied to this letter. Suppose, for example, that we wish to indicate different periods of time; denoting the first by t, the second may be denoted by t', the third by t'', and so on.

The question for Double Fellowship may be stated, in general terms, thus:

A number of persons, A, B, C, enter into partnership. They advance sums denoted by p, p', p'', which they continue in trade for times denoted by t, t', t'', respectively. They gain a sum, a. What is each man's share of the gain?

The letters p, p', p'', denote sums referred to the same unit, as dollars, pounds, &c.; and the letters t, t', t'', denote periods of time referred to the same unit, as days, months, years, &c.

It is evident that each man's share of the gain should depend, 1°, upon his proportion of the stock; 2°, upon the time during which his stock is employed in trade. The shares will be, therefore, as the stock multiplied by the time, or in the compound ratio of both.

Let x, x', x'', represent the shares of the persons, respectively, in the gain; then

$$x : pt :: x' : p't' :: x'' : p''t'';$$

whence $$x + x' + x'' : pt + p't' + p''t'' :: x : pt$$
$$:: x' : p't'$$
$$:: x'' : p''t'';$$

whence, by substitution, &c.,

$$(pt + p't' + p''t'')\,x = apt$$

$$x = \frac{apt}{pt + p't' + p''t''}.$$

To find, therefore, a share of the gain, 1°, *Multiply each man's stock by the time it is in trade, and consider the product as his share of the joint stock.* 2°, *Multiply each man's share in the joint stock by the whole gain, and divide by the sum of the joint stocks.*

The formula for x in this case returns, it is evident, to that for Single Fellowship, if we suppose t', t'' each equal to t. The formula for the latter is, therefore, comprehended in this.

Ex. 1. Three men trade in company. A put in $4000 for 12 months, B put in $3000 for 15 months, and C put in $5000 for 6 months. The whole gain was $615. What were their respective shares?

Ans. A's $240, B's $225, C's $150.

Ex. 2. Two men trade in company. One puts in $1000 for 18 months, the other $1200 for 6 months. They gain $400. What is each man's share of the gain?

Ans. A's $285.72, B's $114.28.

Ex. 3. Four men trade in company. A puts in $900 for 6 months, B $1200 for 9 months, C $1500 for 3 months, D $1800 for 12 months. They gain $850. What is each man's share of the gain?

ANS. A's $108.51, B's $217.02, C's $90.42, D's $434.04.

INTEREST.

67. Interest is the sum paid for the use of money. It is usually reckoned at so much per cent., that is, at so much upon 100, whatever the denomination of money employed. Thus, if $6 are paid for the use of $100 for a year, the interest is said to be 6 per cent. a year.

The sum loaned is called the *principal;* the principal and interest added together is called the *amount.*

The interest of 1, as $1, £1, &c., for a year, is called the *rate.* Thus, at 6 per cent., the rate will be .06; at 5 per cent., it will be .05.

Interest depends, it is evident, upon the rate, and also upon the time. To find the interest, therefore, we multiply the principal by the product of the rate by the time. When interest is paid upon the principal only, it is called *simple interest.*

The following general question will comprehend the different questions respecting interest:

1. What sum of money must be put at interest at a given rate, in order to amount to a given sum in a given time?

Let $p =$ the principal, or sum put at interest,

$r =$ the rate,

$a =$ the given amount,

$t =$ the given time.

Then, by the question, $p + trp = a$,

or $(1 + tr)p = a;$

whence $$p = \frac{a}{1 + tr}.$$

We have, therefore, the following rule by which to find the principal, viz.: *Multiply the rate by the time, and add 1 to the product; divide the amount by the sum thus obtained, and the quotient will be the principal.*

Ex. 1. A man lent a certain sum of money at 6 per cent.; at the end of 5 years he received, for principal and interest, $292.50. What was the sum lent? Ans. $225.

Ex. 2. A man put at interest a certain sum for 4 years, at 7 per cent. At the end of the time, he received, for principal and interest, $1216. What was the sum lent?

Ans. $950.

Ex. 3. A merchant finds that he has gained 20 per cent. upon his capital, and that by this means it has increased to $1890. What was his capital? Ans. $1575.

The equation $p + trp = a$ contains four different things, any one of which may be determined, when the others are known.

2. Let it be required to determine for what time a given principal must be put at interest, in order to amount to a given sum at a given rate. From the equation

$$p + trp = a,$$

we deduce the value of t, which gives

$$t = \frac{a - p}{rp}.$$

From which we derive the following rule: *From the amount subtract the principal, and divide the remainder by the product of the rate multiplied by the principal.*

Ex. 1. A man put at interest $440 at 4 per cent.; at the end of the time, he received, for principal and interest, $510.40. For what time was the money lent?

Ans. 4 years.

Ex. 2. A man put at interest \$450.50 at 6 per cent.; at the end of the time, he received, for principal and interest, \$531.59. How long was his money lent?

ANS. 3 years.

3. Let it be required next to determine at what rate a given principal must be put at interest, in order to amount to a given sum in a given time.

From the equation $p + trp = a$, deducing the value of r, we have

$$r = \frac{a - p}{tp}.$$

Whence, to find the rate: *From the amount we subtract the principal, and divide the remainder by the product of the time by the principal.*

Ex. 1. A gentleman puts out \$125 for 8 years; at the end of the time he received \$155, for principal and interest. At what rate per cent. did he loan his money?

ANS. 3 per cent.

Ex. 2. A man puts at interest \$780 for 10 years; at the end of the time he receives, for principal and interest, \$1170. At what rate per cent. was his money put out?

ANS. 5 per cent.

4. Let it next be required to determine how long a given sum must be kept at interest in order to be doubled, tripled, &c.

Let k denote the number of times the principal is required to be repeated; then, in the general formula $p + trp = a$, a, it is evident, must be equal to kp; whence,

$$p + trp = kp,$$

or

$$1 + tr = k;$$

whence

$$t = \frac{k - 1}{r}.$$

The time, it is evident, is independent of the principal; that is, the time will be the same, whatever the sum put out.

Ex. 1. When will a principal be doubled, at 6 per cent.?

ANS. In $16\frac{2}{3}$ years.

Ex. 2. When will a principal be tripled, at 5 per cent.?

ANS. In 40 years.

DISCOUNT.

68. Discount is an allowance made for the payment of money before it is due.

The *present worth* of a debt due some time hence is a sum that, put at interest, will amount to the debt when it becomes due.

The discount is found by subtracting the present worth from the amount of the debt, or sum due

In the general formula for interest, if we regard a as the sum due, p its present worth, we have

$$p = \frac{a}{1 + tr}.$$

What discount should be made for the present payment of a debt of $246.21, due 2 years and 8 months or $2\frac{2}{3}$ years hence, interest being reckoned at 6 per cent.?

By the formula we obtain for the present worth $212.25. The discount is, therefore, $33.96.

We may, however, make a rule by which to find the discount directly. In order to this, let $d =$ the discount;

then $$d = a - \frac{a}{1 + tr};$$

whence $$d = \frac{tra}{1 + tr}.$$

To find the discount, therefore, *Multiply the debt by the product of the time by the rate; add unity next to the prod-*

uct of the time by the rate; the former quantity, divided by the latter, will be the discount.

Ex. 1. What is the discount of $129.85 for one year, at 6 per cent.? Ans. $7.35.

Ex. 2. How much ready money will purchase a note of $1719.04 due 6 years hence, discounting at 6 per cent.?

Ans. $1264.

Ex. 3. Suppose I owe a note of $560, to be paid in 4 years and 3 months, and wish to pay it now; what must be discounted for present payment? Ans. $113.78 . . .

PROGRESSIONS.

69. When a series of numbers increases or decreases by a constant ratio, the series is called a *progression.*

If the series increase or decrease by the addition or subtraction of the same number, it is called an *arithmetical progression*, or, *progression by difference.*

Thus the numbers 1, 4, 7, 10, form a progression by difference.

The number by which the successive terms of the series increase or decrease is called the *common difference* or *ratio.* Thus, in the series, 1, 4, 7, 10, the ratio is 3.

The series is said to be *increasing* or *decreasing* according as its terms are formed by the successive addition or subtraction of the ratio.

Let $a, b, c, d, \ldots$ be the successive terms of a progression by difference; the progression is indicated thus:

$$\div a \,.\, b \,.\, c \,.\, d \,.\, \ldots\,.$$

2. If the series increase by a common multiplier, or decrease by a common divisor, it is called a *geometrical progression*, or, *progression by quotient:*

The number by which we multiply or divide is called the ratio.

If we pass from one term to the next following by multiplying by the ratio, the series is called an *increasing* progression; if by dividing, it is called a *decreasing* progression.

Thus, of the two following series,

$$3,\ 9,\ 27,\ 81,\ 243$$
$$243,\ 81,\ 27,\ 9,\ 3,$$

the first is an increasing, the second a decreasing progression by quotient. In the first, we pass from one term to the next following by multiplying by 3; in the second, by dividing by 3, or, which is the same thing, by multiplying by $\frac{1}{3}$. In general, it will be most convenient to regard the ratio as a multiplier, either entire or fractional. If the progression is decreasing, the ratio will be a proper fraction, or less than unity.

The ratio is found by dividing any term by that which immediately precedes it. In the first of the series above, the ratio is 3; in the second, it is $\frac{1}{3}$.

Let a, b, c, d . . . be a progression by quotient. The progression is indicated thus:

$$\div\!\div\ a : b : c : d : \ldots$$

The first and last terms of a progression are called the *extremes*; the intermediate ones, the *means*.

PROPERTIES OF PROGRESSIONS BY DIFFERENCE.

70. Let us take the increasing progression by difference

$$\div\ a \,.\, b \,.\, c \,.\, k \,.\, l \ldots$$

Let d be the common difference. Then, from the nature of the progression, we have

$$a = a$$
$$b = a + d$$

$$c = a + 2d$$
$$k = a + 3d$$
$$l = a + 4d.$$

Here the second term, it is evident, is equal to the first + the ratio, the third is equal to the first + *twice* the ratio, the fourth is equal to the first + *three* times the ratio, and so on. And, in general, it will be seen, that any term of the series is equal to the first term + the ratio taken a number of times less one than the number of the term in the series.

Let L be any term of the series, and n the number which marks its place; we shall have, if the series is increasing,

$$L = a + (n - 1)d;$$

if decreasing,
$$L = a - (n - 1)d.$$

Thus, to find any term of the series: *Multiply the ratio by one less than the number of the term*, and if the series is increasing, *add the product to the first term*, or if the series is decreasing, *subtract the product from the first term.*

Ex. 1. What is the 15th term of the decreasing progression, in which the first term is 62 and the ratio 3?

Ans. 20.

Ex. 2. A man bought 50 yards of cloth, for which he was to pay 6 cents for the first yard, 9 cents for the second, and so on, increasing by the common difference 3. How much did he pay for the last yard? Ans. $1.53.

Ex. 3. A man agrees to dig a well 50 feet deep for what the last foot would come to, supposing that he were to receive 75 cents for the first foot, $1.25 for the second, and so on. What did he receive for digging the well?

Ans. $25.25.

71. The expression for L contains four different things, viz., L, a, n and d, any one of which may be found, when the other three are given.

1. What is the formula for n, when the other things are given?

ANS. $n = \frac{L - a}{d} + 1.$

Ex. 1. The extremes of a progression by difference are 2 and 23, and the ratio is 3. What is the number of terms?

ANS. 8.

Ex. 2. A man travelled from a certain place to Boston. He went 7 miles the first day and 52 the last, having increased each day the distance travelled by 5 miles. How many days was he on the road? ANS. 10 days.

2. Let the learner prepare the formula, and solve the following questions:

Ex. 1. A man has 8 sons; the youngest is 4 years old, and the eldest is 32, their ages increasing in arithmetical progression. What is the common difference of their ages?

ANS. 4 years.

Ex. 2. A traveller is 13 days upon his journey. He travels 14 miles the first day and 50 the last, increasing the distance travelled each day by the same number of miles. What was the daily increase? ANS. 3 miles.

3. State the rules derived from the formulas in the preceding numbers.

72. Let us resume the general progression

$$\div\, a \,.\, b \,.\, c \,.\, d \,.\,.\,.\,.\, l.$$

This returns to $\div\, a \,.\, (a + d) \,.\, (a + 2\,d) \,.\,.\,.\,.\, l$. Putting S for the sum of all the terms, and writing the series under itself in an inverse order, we have

$$S = a + (a + d) + (a + 2\,d) + \ldots\, l$$

$$S = l + (l - d) + (l - 2\,d) + \ldots\, a;$$

adding both,

$$2\,S = (a + l) + (a + l) + (a + l) + \ldots . (a + l):$$

or, taking n terms, $2\,S = n\,(a + l)$;

whence, $$S = \frac{n\,(a + l)}{2}. \quad (1)$$

To find, therefore, the sum of all the terms, when the extremes and the number of terms are given, *Multiply the sum of the extremes by the number of terms, and take one-half of the product.*

Ex. 1. In a progression by difference, the first term is 5, the last term 62, and the number of terms 20. What is the sum of the series? Ans. 670.

Ex. 2. How many strokes does the hammer of a clock strike in 12 hours? Ans. 78.

Ex. 3. If 100 eggs were placed in a right line, exactly one yard from each other, and the first one yard from a basket, what distance will a man travel who gathers them up singly and places them in the basket?

Ans. 5 miles, 1300 yds.

Ex. 4. A student purchased a lot of books, amounting in all to 50. The prices were in arithmetical progression. For the lowest price he paid 25 cents, for the highest $1.70. How much did his books cost him? Ans. $48.75.

73. The equation $2\,S = n\,(a + l)$ contains four quantities, viz., S, n, a and l, any one of which may be found when the other three are given.

1. Let it be required to find a formula for n, the other things being given; we have

$$n = \frac{2\,S}{a + l}.$$

Ex. 1. The first term of a progression by difference is 3, the last term 41, and the sum of the series 440. What is the number of terms? Ans. 20.

Ex. 2. A farmer set out 580 trees in his orchard, so that the successive rows formed a progression by difference. In the first row he put 10, and in the last 48. How many rows were there? ANS. 20.

2. Let the learner prepare formulas for a and l, and solve the following examples:

Ex. 1. The sum of a progression by difference is 670, the number of terms 20, and the last term 62. What is the first term? ANS. 5.

Ex. 2. A traveller who had been 25 days on the road, found that he had made 600 miles. The distances travelled each day were in arithmetical progression, and he had travelled the last day 44 miles. How many miles did he travel the first day? ANS. 4 miles.

Ex. 3. The sum of a progression by difference is 2385, the first term is 7, and the number of terms 20. What is the last term? ANS. $231\frac{1}{2}$.

Ex. 4. A car runs down an inclined plane of 1500 feet in 40 seconds, its speed increasing each second in arithmetical progression. It ran 10 feet the first second. How many feet did it run the last? ANS. 65 feet.

74. The formulas

$$L = a + (n - 1)\, d, \quad S = \tfrac{1}{2}\, n\, (a + l)$$

comprehend all the cases which occur in arithmetical progression.

1. If 75 acres of land are sold at the rate of 15 cents for the first acre, 20 for the second, 25 for the third, and so on, what is the whole amount of the sale?

In this question we have given the first term, the number of terms, and the ratio to find the sum of all the terms. In order to this, from the formula for L we find first what the last acre cost, or the last term of the series. This is

385 cents. By means next of the formula for S, we obtain \$150.00, the sum for which the whole 75 acres were sold.

If we substitute for l in the second formula its value in the first, we have

$$S = \frac{n\,[2\,a + (n-1)\,d]}{2} = n\,a + \frac{n\,(n-1)\,d}{2},$$

from which the value of S, or the answer required, is obtained directly from the data given in the question.

Ex. 1. A man being anxious to purchase a horse, offers 1 dollar for the first nail in his shoes, 4 for the second, and so on, in arithmetical progression. Now, there being 8 nails in each shoe, what will the horse cost him?

ANS. \$1520.

Ex. 2. A gentleman purchased 80 yards of cloth, giving 6 cents for the first yard, 10 for the second, and so on, in arithmetical progression. What did the cloth cost him?

ANS. \$131.20.

Ex. 3. A car, descending an inclined plane, moves 5 feet the first second, 15 the second, 25 the third, and so on, increasing 10 feet every second. How far will it move in a minute? ANS. 18000 feet, or $3\frac{9}{22}$ miles.

Ex. 4. A laborer agreed to dig a well 40 feet deep, for which he was to be paid as follows: 30 cents for the first foot, 50 for the second, and so on, increasing 20 cents for each subsequent foot. What did he receive for the whole job? ANS. \$168.00.

Ex. 5. The resistance of the air not being considered, a heavy body would fall, in the air, through a space of 16.1 feet in the first second, 48.3 feet in the next, and so on, increasing 32.2 feet each succeeding second. Through how many feet would it fall in 10 seconds? ANS. 1610.

Ex. 6. A gentleman hires a servant, and promises him for the first year only $30 in wages, but for each following year $4½ more than for the preceding. The servant remained with him 17 years. What was the whole amount of his wages? ANS. $1122.

2. In a company, the conversation turning on domestic economy, a person said, "This year I have saved $78, and since I first entered into employment until now, I have saved of my salary each year $2 more than in the preceding." The person was in office 25 years. What was the whole amount of his savings?

In this example we have given the last term, the number of terms, and the ratio, to find the sum of all the terms, or l, n and d, to find S.

To make a formula for this case, we eliminate a, which is not one of the given or required things, from the two equations, $L = a + (n - 1)\,d$, $S = \frac{1}{2}\,n\,(a + l)$. Finding, for this purpose, the value of a in each, and equating the results, we have

$$\frac{2\,S - n\,l}{n} = l - n\,d + d,$$

an equation which contains the three given things, n, l and d, and the required thing, S. Deducing the value of this last, we have for the formula required

$$S = n\,l - \frac{n\,(n\,d - d)}{2}. \quad (3)$$

From which we obtain, for the answer to the proposed question, $1350.

Ex. 1. A person was condemned by the judge to pay a fine in 16 different installments, each installment to be $4 more than the preceding, and the last to be $80. What was the amount of the fine? ANS. $800.

Ex. 2. In a foundery, a person saw 15 rows of cannon balls, placed one above the other, each row, from the first to the last, containing 20 balls less than the one immediately below it. In the lowest row there were 420 balls. How many were there in the whole pile? Ans. 4200.

3. A debtor, being unable to pay his debt at once, agrees with his creditor to discharge it by monthly installments, paying each month \$50 more than in the preceding one. The first installment was \$600, and the last \$1250. What was the amount of the debt?

In this example, we have given a, d and l, to find S. Eliminating n from the two fundamental equations,

$$L = a + (n - 1)\,d, \quad S = \tfrac{1}{2}\,n\,(a + l),$$

we obtain an equation which contains the three given things, a, d and l, and the required thing, S; and from which we obtain

$$S = \frac{l^2 - a^2}{2d} + \frac{a + l}{2}. \quad (4)$$

Applying this to the proposed question, we obtain \$12950 for the answer.

Ex. 1. A traveller, who wishes to be at the place of his destination in a certain number of days, expedites his journey so much that he goes each day 3 miles more than the preceding. He travels the first day 15 miles, and the last 72 miles. How many miles did his whole journey amount to?

Ans. 870 miles.

Ex. 2. A rocket, as it rises in the air, loses 5 feet of its velocity, or the space it passes through, each second. Its velocity the first second was 70 feet, and the last 10 feet. The time of its ascent was 13 seconds; how high did it rise? Ans. 520 feet.

75. The formula for L, with those now obtained for S, comprise all the different cases which occur, in which three of the things in a progression by difference being given, a fourth is required.

These different cases are, it is easy to sec, 20 in number. Four of them, as, for example, when a, d and S are given to find n, involving equations of a higher degree than those we have thus far considered, we are not now prepared to solve. The learner may, however, prepare the formulas, and solve the following particular cases:

Ex. 1. The sum of all the terms in a progression by difference is 155, the last term is 10, and the number of terms 30. What is the common difference? Ans. $\frac{1}{3}$.

Ex. 2. A merchant has $2535 to pay by 15 installments. The first one was $50, but each succeeding one was invariably greater than the one preceding, so that the last installment was $288. What was the common difference between each successive payment? Ans. $17.

Ex. 3. In a progression by difference, the sum of all the terms is 450, the last term is 50, and the number of terms 15. What is the common difference? Ans. $2\frac{6}{7}$.

Ex. 4. A father, dying, bequeathed his property, amounting to $41400, to his 9 sons. To the youngest he gave $5200, and to the others less, decreasing regularly by a constant difference. What was this constant difference?

Ans. $150.

Ex. 5. What is the first term of an arithmetical progression, the number of terms being 100, the common difference 3, and the sum of all the terms 15350? Ans. 5.

Ex. 6. A stone is thrown vertically upward, with a force which causes it to pass through 180 feet the first second. It rises with a velocity decreasing by the same quantity for 12 seconds, when it has reached the height of 1170 feet. Through what space did it pass during the last second of its ascent? Ans. 15 feet.

76. We close with the following problem. Between the numbers 7 and 106 it is required to insert 10 mean terms, so that, with these, they shall form a progression by difference.

In order to solve the problem, it is necessary to find the common difference of the terms. The formula for L gives for this purpose

$$d = \frac{L - a}{n - 1},$$

from which we obtain 9 for the common difference; whence the series will be easily found.

Ex. Insert between the numbers 3 and 32 six mean terms.

PROPERTIES OF PROGRESSIONS BY QUOTIENT.

77. Resuming the general progression $\div\div a : b : c : d \ldots$, and putting $q =$ the ratio, we have

$$b = a q$$
$$c = b q = a q^2$$
$$d = c q = a q^3.$$

From which we learn, that to find any term of the series, *we multiply the first term by the ratio raised to a power one less than the number of the terms.*

Let L represent any term of the series; the formula corresponding to the rule will be

$$L = a q^{n-1}.$$

Ex. 1. What is the 8th term of the progression 2 : 4 : 8 . . . ? ANS. 256.

Ex. 2. Required the last term of a progression by quotient, of which the first term is 5, the ratio 2, and the number of terms 8. ANS. 640.

Ex. 3. A gentleman, dying, left 9 sons, and bequeathed his estate in the following manner: to his executors £50: his youngest son to have twice as much as the executors, and each son to have double the amount of the son next younger. What was the eldest son's portion? Ans. £25600.

The expression for L contains four things, L, a, q and n, any three of which may be given to find the fourth.

The formula will be easily obtained for the following case:

Ex. 1. The last term of a progression by quotient is 1536, the ratio 2 and the number of terms 10. What is the first term? Ans. 3.

Ex. 2. The last term of a progression by quotient is $\frac{1}{4}$, the ratio $\frac{1}{2}$, and the number of terms 8. What is the first term? Ans. 32.

78. Returning to the general progression

$$\div\div\, a : b : c : d : \ldots k : l,$$

we have, as above,

$$b = a\,q, c = b\,q, d = c\,q, \ldots l = k\,q;$$

adding these equations, member to member,

$$b + c + d + \ldots l = (a + b + c + \ldots k)\,q. \quad (1)$$

Let S represent the sum of all the terms; then

$$b + c + d + \ldots l = S - a$$
$$a + b + c + \ldots k = S - l;$$

Whence by substitution in equation (1)

$$S - a = q\,(S - l)$$

and

$$S = \frac{q\,l - a}{q - 1}.$$

We have, therefore, the following rule, by which to find the sum of any number of the terms of a progression by quotient: *Multiply the last term by the ratio, subtract the* *first term from this product, and divide the remainder by the ratio diminished by unity.*

Ex. 1. The first term of a progression by quotient is 2, the ratio 3, and the last term 4374; what is the sum of all the terms? ANS. 6560.

Ex. 2. The first term of a progression by quotient is 7, the ratio $\frac{1}{2}$, and the last term $1\frac{3}{4}$; what is the sum of all the terms? ANS. $12\frac{1}{4}$.

Ex. 3. If I discharge a debt by paying 1 dollar the first month, 4 the second, and so on, in a four fold ratio, the last payment being $65536, — what was the whole debt?

ANS. $87381.

Ex. 4. A man bought a number of bushels of wheat on the condition that he should pay 1 cent for the first bushel, 3 for the second, 9 for the third, and so on, to the last, for which he paid $196.83. What did he pay for the whole?

ANS. $295.24.

79. The expression for S contains four things, S, q, l and a, any three of which may be given to find the fourth.

1. If l be required, the other things being given, we have

$$l = S - \frac{S - a}{q}.$$

Ex. 1. The sum of a progression by quotient is 2343, the first term 3, and the ratio 5; what is the last term?

ANS. 1875.

2. What is the formula for a, the other things being given?

Ex. What is the first term of a progression by quotient, the sum of which is 1785, the last term 896, and the ratio 2? ANS. 7.

3. What is the formula for q, the other things being given?

Ex. The sum of a progression by quotient is 27305, the last term 20480, the first term 5; what is the ratio?

ANS. 4

80. From the fundamental equations

$$L = a\,q^{n-1}, \quad S = \frac{q\,l - a}{q - 1}$$

formulas may be derived for all the variations of the data, which can occur. But these, in general, involve equations of a higher degree than those we have thus far considered. We shall take only the following case.

What is the formula for S, when a, q, and n, are the given things?

ANS. $S = \dfrac{a\,(q^n - 1)}{q - 1}$.

Ex. 1. Required the sum of 11 terms of the progression 3 : 9 : 27 . . . . ANS. 265719.

Ex. 2. A man sold 15 yards of cloth; the first yard for 1 shilling, the second for 2, the third for 4, and so on. For how much did he sell the whole? ANS. £1638 7*s.*

Ex. 3. A gentleman, without reflecting upon the result, agreed to pay his gardener 1 dollar for the first month, 2 for the second, and so on, doubling his wages each month for a year. What would be the amount of the year's wages?

ANS. $4095.

Ex. 4. If the human race, after making a proper deduction for those who died, had doubled every 20 years, how many of the descendants of Adam would have been living when he was 500 years old? ANS. 33554430.

POSITION.

81. This name is given, in arithmetic, to a rule by which many difficult questions may be solved, by assuming one or more numbers for the answer, and working upon them as if they were the true answer.

Let it be proposed, for example, to find a number such, that its half, fourth and fifth parts may together be equal to 76.

Let us suppose that 100 is the number. Working with this as if it were the true number, we find that its half, fourth and fifth parts are together equal to 95, a result too great by 19; 100 is not, therefore, the right number.

The rule of position teaches how to find the true number by means of the operations we have performed upon this supposed number.

To demonstrate the rule, let x be the true answer to the proposed question. Let the operations required to be performed upon x be such, that in their aggregate we shall have a times x, and for the result of these operations a number b; or in other words, suppose that from the conditions of the question we derive the equation

$$a x = b, \quad (1)$$

x being the true answer, and a and b known numbers.

Let us assume a number, x', to be the answer, and performing upon this the same operations that we have performed upon x, let the result be b'; x' will not be the true answer, since the result is different from b. We shall have, nevertheless, the equation

$$a x' = b'. \quad (2)$$

Comparing the equations (1) and (2), we have

$$a x : a x' :: b : b';$$

whence $$b' : b :: x' : x.$$

That is, *the false result is to the true, or that given in the question, as the false or supposed number is to the true or required number.*

Finding the value of x from this proportion, we have

$$x = \frac{x' b}{b'}.$$

82. If one assumed number only is employed, the process is called *Single Position.* We have, therefore, the following rule for Single Position:

1°. *Assume any convenient number and perform upon it the operations required by the conditions of the question.*

2°. *Multiply the assumed number by the true result and divide the product by the result obtained with the assumed number.*

Applying the rule to the question proposed, we have

$$\frac{100 \times 76}{95} = 80,$$

the true number, or answer.

This rule will apply to all questions which give an equation of the form $a x = b$, that is, in which x does not occur in one member, and is a factor of all the terms of the other. The proposed is evidently of this class, since, putting x for the number sought, it gives the equation

$$\frac{x}{2} + \frac{x}{4} + \frac{x}{5} = 76,$$

or $$(20 + 10 + 8)\, x = 3040.$$

The rule will, therefore, apply to all questions in which the required number is increased or diminished by any of its parts or multiples, either by addition, subtraction, multiplication or division.

Ex. 1. A man bought a horse, chaise and harness for \$216. The horse cost twice as much as the harness, and the harness one-third as much as the chaise. What was the cost of the chaise? Ans. \$108.

Ex. 2. A schoolmaster, being asked how many scholars he had, replied, if he had as many more, $\frac{1}{2}$ and $\frac{1}{4}$ as many more, he would have 88. How many had he?

ANS. 32.

Ex. 3. A commission merchant received $2\frac{1}{2}$ per cent. for the sale of an invoice of merchandise. What was the amount of the invoice, the total amount of the sale and commission being $1666.24? ANS. $1625.60.

Ex. 4. A man performed a journey of 630 miles, going twice as far the second day as on the first, and three times as far the third day as on the second. How far did he travel each day? ANS. 70 miles the first day, &c.

Ex. 5. A man, going to market, was met by another, who said, "Good morrow, neighbor, with your hundred geese." He replied, "I have not a hundred; but if I had as many more, and half as many more, and two geese and a half besides, I should have a hundred." How many had he?

ANS. 39.

Ex. 6. If 18 per cent. is lost by selling merchandise at $2050, at what price should it have been sold to lose only 10 per cent.? ANS. $2250.

Ex. 7. A and B, having found a purse of money, disputed who should have it. A said that $\frac{1}{5}$, $\frac{1}{10}$ and $\frac{1}{20}$ of it amounted to $35, and if B would tell him how much was in it, he should have the whole; otherwise he should have nothing How much did the purse contain?

ANS. $100.

Ex. 8. A laborer received $1.50 for every day he worked, and lost 50 cents every day he was idle. He worked twice as many days as he was idle, and at the end of the time he received $42.50. How many days did he work?

ANS. 34 days.

83. The question sometimes requires that two assumed numbers should be employed. The process in this case is called *Double Position.*

To demonstrate the rule, let it be supposed that x is the true answer, and that the operations required by the question lead to an equation of the form

$$a x + b = m.$$

By transposition, this equation becomes

$$a x + b - m = 0. \quad (1)$$

Let us now assume a number x', and performing upon x' the same operations that are performed upon x, suppose we obtain $$a x' + b - m = e; \quad (2)$$
x' cannot be the answer, since, e being different from 0, the operations performed upon it lead to a false result.

Suppose, again, another number, x'', and that performing upon it the same operations as upon x, we have

$$a x'' + b - m = e'; \quad (3)$$

x'' cannot be the answer, since the result, e', differing from 0, is also a false result.

This being done, equation (1), subtracted from (2), gives

$$a\,(x' - x) = e; \quad (4)$$

subtracted from (3), it gives

$$a\,(x'' - x) = e'. \quad (5)$$

Dividing (4) by (5), and striking out the common factor a

$$\frac{x' - x}{x'' - x} = \frac{e}{e'}.$$

From which we obtain $$x = \frac{e x'' - e' x'}{e - e'}.$$

We have supposed the errors e, e' both alike, that is, both + or both —. If they are unlike, that is, if one is + and the other —, the formula for x will then be

$$x = \frac{e x'' + e' x'}{e + e'}.$$

From which we deduce the following rule for Double Position:

1°. *Assume two convenient numbers, and proceed with them according to the conditions of the question, and note the error.*

2°. *Multiply the second supposed number by the first error, and the first supposed number by the second error.*

3°. *If the errors are alike, that is, if both are too great or both too small, divide the difference of the products by the difference of the errors.*

4°. *But if the errors are unlike, that is, one too great and the other too small, divide the sum of the products by the sum of the errors.*

Ex. 1. Five times a certain number increased by 12 is equivalent to 7 times the number, diminished by 20. What is the number?

Suppose that it is 20. Performing the operations indicated in the question, the error will be $+8$; suppose next that it is 18, the error will then be $+4$; whence by the rule,

$$\frac{18 \times 8 - 20 \times 4}{8 - 4} = 16, \text{ the answer.}$$

Ex. 2. Two men have the same income. A saves $\frac{1}{5}$ of his, but B spends \$325 per year more than A, and at the end of 5 years finds himself \$625 in debt. What is the annual income of each?

First suppose \$800, the error will be $+200$; next suppose \$1200, the error will be -200; whence

$$\frac{800 \times 200 + 1200 \times 200}{400} = \$1000, \text{ the answer.}$$

Ex. 3. A man being asked, in the afternoon, what o'clock it was, answered that the time past from noon was equal to $\frac{1}{8}$ of the time to midnight. Required the time.

Ans. 20 minutes past 1 o'clock.

Ex. 4. What number is that whose half is as much less than 75, as its double is greater than 94? ANS. $67\frac{3}{5}$.

Ex. 5. A man divided a certain sum of money equally between his son and daughter; but had he given his son 33 dollars more, and his daughter 47 dollars less, her share would have been but $\frac{1}{3}$ of his. What was the sum divided?

ANS. $174.

Ex. 6. What number is that, which being multiplied by 4, and 30 subtracted from the product, and being divided by 4, and 30 added to the quotient, the sum and difference shall be equal? ANS. 16.

84. The errors being found as in the preceding examples a correction may be applied to either of the assumed numbers, so as to obtain the required number immediately from it. Thus in the formula above, e being one of the errors and x' the assumed number from which it is derived, let x be less than x. Putting y for the correction to be applied to x', we have

$$x' + y = x = \frac{e\,x'' - e'\,x'}{e - e'};$$

from which we obtain $$y = \frac{(x'' - x')\,e}{e - e'};$$

or if x' be greater than x,

$$y = \frac{(x' - x'')\,e}{e - e'}.$$

Having obtained this formula for the correction of one of the assumed numbers, we have the following rule for Double Position:

1°. *Assume two convenient numbers and perform upon each the operations required in the question.*

2°. *Note the errors of the results and mark each of them with the sign + or —, according as it is in excess or defect.*

3°. *Multiply the difference of the assumed numbers by* *either of the errors, and divide by the difference of the errors if they are alike, or by their sum if they are unlike; the quotient will be a correction to be applied to the assumed number which produced the error by which we multiply.*

4°. *If the assumed number is greater than the true, the correction subtracted from it, or if it be less, the correction added to it, will give the true number.*

We give a few additional examples.

Ex. 1. A man agreed to carry 20 earthen vessels to a certain place on this condition; that for every one delivered safe he should receive 11 cents, and for every one he broke he should forfeit 13 cents; he received 124 cents. How many did he break? Ans. 4.

Ex. 2. A man being asked what his carriage cost, replied, "If it had cost twice as much as it did, and $20 more, it would have cost $370." What was the cost of the carriage?

Ans. $175.

Ex. 3. Three persons have coins of the same kind and value; the second has twice as many as the first, and 4 more; the third as many as the first and second together, and 6 more; and the whole number is 44; how many had the first? Ans. 5.

Ex. 4. A merchant increases his capital yearly by 20 per cent., but takes from it every year $1000 for the support of himself and family. After he had carried on his business, in this manner, for 3 years, he finds, after deducting the usual sum of $1000, that his capital has increased $200 more than $\frac{1}{5}$ of the original sum. What was his original capital?

Ans. $30000.

Ex. 5. A man, to please his children, brings home a number of apples, and divides them as follows: to the first and eldest he gives half of the whole number, less 8; to the second, the half of the remainder, again diminished by 8; and he does the same with the third and fourth. After this, he gives the 20 remaining apples to the fifth. How many apples did he bring home? ANS. 80.

Ex. 6. Two persons go to a fair, and the first says to the second, How many ducats have you? The second answered and said, If I had 30 of yours, I should have as many as you; and the first answered and said, If I had 20 of yours, I should have twice as many as you; how many ducats had each of them? ANS. The first 180, the second 120.

The rule of Double Position is applicable to all questions which can be solved by Single Position, and, in general, to most questions that can be solved by algebraic equations. Both rules admit of important applications, and are, therefore, deserving consideration. In general, however, the questions performed by them can best be performed by the aid of algebra. By a comparison of the two methods in the preceding examples, the learner will easily see the great superiority of the direct and simple methods of algebra, over the indirect and often complicated methods or expedients of arithmetic.

SQUARE ROOT OF NUMBERS.

85. By the square root of a number is meant a number such that, if multiplied by itself, it will produce the given number. Thus the square root of 81 is 9, since 9 multiplied by itself gives 81.

The square root of a number is indicated by the sign $\sqrt{}$ placed before it; thus $\sqrt{}$ 53, means square root of 53.

A number whose square root can be found exactly, is called a *square number*, or perfect square.

If a number were for the first time presented to us to find its square root, we should endeavor to accomplish the object by guessing some number near to it, and then, by trial, ascertaining whether its square were equal to the proposed; correcting it, if necessary, until, by repeated trials, the true root is found. The process for finding the square root is one substantially of this kind. By the rule, the successive trials are reduced to within the narrowest limits.

Before proceeding to large numbers, it is necessary to know the roots of some of the small numbers. Below is a table of the first nine numbers, with their squares, or second powers, written beneath them.

1	2	3	4	5	6	7	8	9
1	4	9	16	25	36	49	64	81

Since the second line contains the squares of the numbers in the first; conversely, the numbers in the first line are the square roots of those in the second.

86. Let it now be proposed to find the square root of 4624.

The root, it is evident, cannot have less than two places of figures, for the square of 10, the least number consisting of two figures, is 100, and 4624 is manifestly greater than this. But again, it cannot have three figures, for the square of 100, the least number consisting of three figures, is 10000, and 4624 is manifestly less than this. The root of 4624 will necessarily, therefore, have two places of figures, viz., units and tens, and two only.

To proceed with the investigation of the rule, let us now observe the manner in which the units and tens of a number are employed in forming its square, since in this way we may learn how to return from the number to the units and tens of its root.

Let the number be 57. This may be decomposed into 50 or 5 tens, and 7 units. Let a represent the tens and b the units, then

$$(a + b)^2 = a^2 + 2ab + b^2$$

That is, the square of a number consisting of units and tens, is equal *to the square of the tens + twice the product of the tens by the units + the square of the units.*

Forming the square of 57 by the rule, we have

The square of the tens, a^2,	$= 2500$
Twice the tens by the units, $2ab$,	$= 700$
The square of the units, b^2,	$= 49$
	$(57)^2 = 3249$

If, now, 3249 were the proposed number whose root is required, and we could decompose it into the three parts above, the root would be easily found. For, to obtain the tens, we should have only to take the root of the first part, 2500, which we know to be 50, or 5 tens; and this being found, the units would readily be found from either of the other two parts.

Let us now see if we can decompose the proposed 4624 into the same three parts of which it is also made up. This we cannot do exactly, but we can come sufficiently near to it for our purpose.

Indeed, from what has been done, as well as from the nature of the case, it is evident, that the square of the tens can have no significant figure less than hundreds. We then set aside the two right hand figures of the proposed as forming no part of the square of the tens, which we do by a period, thus, 46.24. And we now see that 4600 will be the square of the tens, either exactly, or something more, in consequence of the units carried to it from the other parts By taking, then, the root of 46, we shall obtain the significant figure of the tens, or this figure within a unit. Looking at the table above, we see that 46 is not a square number, and that the nearest square to it is 36, the root of which is 6 We take, then, 6 for the significant figure of the tens of the root, and which we call either 6 tens or 60 units.

Subtracting next the square of 6 tens from the proposed, 4624, the remainder will be 1024, and this must be equal to the two remaining parts of the square, viz., to *twice the product of the tens by the units + the square of the units.* Can we separate these two parts? We cannot exactly, but still sufficiently near for our purpose. Indeed, we know that the part, twice the product of the tens by the units, can have no significant figure less than 10; the 4, then, will form no part of this product, and we separate it from the rest by a period, thus, 102.4. The 102, therefore, will contain twice the product of the significant figure of the tens by the units, and something more arising from the units carried to it from the remaining part, *the square of the units.* If, then, we divide 102 by 2×6, or 12, we have 8, which will be the unit figure, either exactly, or within 1 or 2 units. We try 8; twice the tens, or 12 multiplied by 8, gives 96 tens, or 960 units, and the square of 8 is 64; these together make 1024, the remainder of the proposed after taking from it the square of the tens of the root before found.

Thus we find that the square root of 4624 is 68. Recomposing the square from the root, we have

The square of the tens, a^2,	$= 3600.$
Twice the tens by the units, $2ab$,	$= 960.$
The square of the units, b^2,	$= 64.$
	$(68)^2 = 4624.$

Let the learner now repeat this process by extracting the roots of the following numbers:

Ex. 1. To find the square root of 625.

Ex. 2. To find the square root of 441.

Ex. 3. To find the square root of 5776.

Ex. 4. To find the square root of 7921.

87. Let it be proposed next to find the square root of 143641. The root will consist of three places, since the proposed is greater than 10000, the square of 100, and less than 1000000, the square of 1000.

Any number, however large, may be considered as composed of units and tens; thus 475 may be considered as composed of 47 tens and 5 units.

Regarding the root of the proposed as composed of units and tens, we set aside, as before, the two right-hand figures, since they form no part of the square of the tens.

The tens of the root will then be found by extracting the root of the remaining part, viz., 1436. This we may do by the process above, precisely as if 1436 were a separate number. Performing the operation, we obtain 37 for the root, with a remainder 67. We shall have, then, 37 tens in the root of the proposed. To find the units, we place 41, the part separated from the rest in the proposed, by the side of the remainder 67, which gives 6741 for the remainder of the proposed, after taking from it the square of the tens of the root. Separating, as before, the 1, and dividing the remaining part, 674, by 74, twice the tens of the root sought, we obtain 9 for the unit figure of the root; which, on trial, proves to be the true unit figure. We have then, 379 for the root sought.

The calculations may be disposed as follows:

```
14.36.41|379
 9      |
 53,6|67
 46 9|
   674,1|749
   674 1|
```

The same process, it is easy to see, may be extended to any number, however large. Hence, the following rule for the extraction of the square root:

1°. *Begin at the right, and separate the number into periods of two figures each. The left-hand period may contain one or two figures.*

2°. *Find the greatest square in the left-hand period, write the root as a quotient in division, and subtract the square from the left-hand period.*

3°. *To the right of the remainder bring down the next period, to form a dividend, and place by its side double the root already found, for a divisor. Seek how many times the divisor is contained in the dividend, rejecting the right-hand figure. Place the quotient in the root and also at the right-hand of the divisor. Multiply the divisor thus increased by the last figure of the root, and subtract the product from the whole dividend.*

4°. *Bring down to the right of the remainder the next period, to form a new dividend. Double the root already found, for a divisor, and proceed as before to find the third figure of the root, and so on.*

Instead of placing points between the figures, it will be more convenient to mark the periods by accents, thus, 14′36′41.

If the dividend will not contain the divisor, the right-hand figure of the former being rejected, a zero must be placed in the root and also at the right of the divisor, and then the next period brought down.

Ex. 1. To find the square root of 21025.

ANS. 145.

Ex. 2. To find the square root of 59536.

ANS. 244.

Ex. 3. To find the square root of 85673536.

ANS. 9256.

Ex. 4. To find the square root of 36372961.

ANS. 6031.

Ex. 5. To find the square root of 3327097761.

ANS. 57681.

If the proposed is not a perfect square, the rule will give the root of the greatest square number next below it. Thus, if it be required to extract the root of 129, the root found by the rule will be 11, with a remainder 8, 121 being the greatest square number next below 129.

88. The root of 129 may be found nearer by approximation with decimals. Since the square root of 100 is 10, the square root of 100 a will be 10 $\sqrt{a}$, or ten times the square root of a. Multiplying 129 by 100, we have 12900, the root of which is 10 times too large for that of 129. But the root of 12900, or rather of the greatest square next below it, is 113; whence we shall have 11.3 for the root of 129, and which is a nearer approach to the true root of this number than 11. If we wish a still nearer root, we add four zeros to 129, which gives 1290000. The root nearest to this is 1135, which is 100 times larger than the root of 129. We shall have, therefore, 11.35 for a still nearer approach to the root of 129. By adding two more zeros, a still nearer approximation may be obtained, and so on.

The following addition may, then, be made to the rule for the square root:

When there is a remainder over, after the last period is brought down, add two zeros to it, and proceed as before, placing the decimal point before the digit next found.

Ex. 1. Find the approximate root of 67321.

Ans. 259.46 +.

Ex. 2. Find the approximate root of 21027.

Ans. 145.006 +.

Ex. 3. Find the approximate root of 153.

Ans. 12.36931 +.

Ex. 4. Find the approximate root of 2268741.

Ans. 1506.23 +.

Ex. 5. Find the approximate root of 25289.

Ans. 159.02 +.

89. The process for finding the square root of a decimal is founded upon the same principle. Let it be required, for example, to find the square root of .0625. Multiplying this by 10000, the square of 100, it becomes 625, the root of which is 25, 100 times too large for that of the proposed. The root of .0625 will be, therefore, .25.

To find the root of a decimal, therefore, beginning at the decimal point, we divide it, as in whole numbers, into periods of two figures each, annexing a cypher, if necessary, to make the decimal places even. We then extract the root as if it were a whole number, observing that the number of decimal places in the root will be one-half the number in the proposed, as it stands after the decimal places are made even.

The case will present no difficulty if the proposed consists of an entire and decimal part. We shall merely add some examples of each.

Ex. 1. To find the square root of .2116.

Ans. .46.

Ex. 2. To find the square root of .0841.

Ans. .29.

Ex. 3. To find the square root of .001024.

Ans. .032.

Ex. 4. To find the square root of 10.4976.

Ans. 3.24.

Ex. 5. To find the square root of 336.234.

Ans. 18.333 +.

Ex. 6. To find the square root of 6842.72340.

Ans. 82.7207 +.

Ex. 7. To find the square root of $8\frac{15}{49}$.

Ans. 2.88203 +.

90. Since the square of a fraction is equal to the square of the numerator divided by the square of the denominator, conversely, the square root of a fraction is equal to the square root of the numerator divided by the square root of the denominator.

If the proposed, then, is a vulgar fraction, we find its root by dividing the square root of the numerator by the square root of the denominator, or, which in general is the better method, we reduce the proposed fraction to a decimal, and then extract the root.

Ex. 1. To find the square root of $\frac{25}{49}$. Ans. $\frac{5}{7}$.

Ex. 2. To find the square root of $\frac{64}{81}$. Ans. $\frac{8}{9}$.

Ex. 3. To find the square root of $\frac{3}{5}$.

Ans. 0.7745 +.

Ex. 4. To find the square root of $\frac{5}{12}$.

Ans. 0.64549 +.

Ex. 5. To find the square root of $2\frac{13}{15}$.

Ans. 1.69331 +.

ALLIGATION.

91. We shall consider here only the rule for alligation medial, the object of which is to determine the mean value of a compound, when the several ingredients and their respective values are given.

For example, a wine merchant bought several kinds of wine, as follows: 100 gallons at 40 cents per gallon; 80 gallons at 60 cents, and 200 gallons at 75 cents per gallon; and mixed them together. It is required to find the cost of a gallon of the mixture?

To prepare the rule, let the ingredients be represented by a, b, c, ... and their prices by a', b', c', ... respectively. Let $x =$ the mean value of the compound. Then the sum of the ingredients multiplied by the mean value, should be equal, it is evident, to the sum of the ingredients multiplied each into its given value. Thus we have

$$x(a + b + c\,..) = a\,a' + b\,b' + c\,c' \ldots$$

and

$$x = \frac{a\,a' + b\,b' + c\,c'\,..}{a + b + c\,..}.$$

Whence, to find the mean value of the compound: *Multiply each ingredient by its given value, add these several products together, and divide the sum by the sum of the ingredients.*

The rule applied to the proposed question gives 62.6 cents + for the answer.

Ex. 1. A goldsmith melted together 12 lbs. of gold 21 carats fine; 8 lbs. 20 carats fine; 9 lbs. 22 carats fine, and 7 lbs. 18 carats fine. Of what fineness is the mixture?

ANS. $20\frac{1}{4}$ carats.

Ex. 2. A farmer mixes 12 bushels of wheat at $1.75 a bushel; 8 bushels of rye at $1, and 6 bushels of corn at 80 cts. a bushel. What is a bushel of the mixture worth?

ANS. $1.30.

Ex. 3. On a certain day the mercury in the thermometer was observed to stand at the following heights: from 6 in the morning to 9, at 64°; from 9 to 12, at 74°; from 12 to 3, at 84°; and from 3 to 6, at 70°. What was the mean temperature of the day? ANS. 73°.

PERMUTATION.

92. The only remaining rule we shall here consider, is that for permutation. By this rule we are taught how to find the number of changes which may be made in the order in which a given number of things may be arranged.

Let there be two things, the letters a and b, for example. These, it is evident, may be placed in two positions in respect to each other, and in two only, viz., $a\,b$ and $b\,a$.

Let there next be three letters, a, b and c. Setting apart the c, the two remaining letters admit as before of two permutations, viz., $a\,b$ and $b\,a$. But the reserved letter c, it is evident, may be placed at the left, between, and at the right of each of these permutations. Thus, *the permutations of three letters will be equal to the permutations of two letters multiplied by three.*

In like manner, if there are four letters, a, b, c, d, the d, it is evident, can have four positions with each one of the permutations of three letters; *the permutations of four letters will, therefore, be equal to the permutations of three letters multiplied by four.*

In general, if there are n letters, there will be $n - 1$ letters in each of the preceding permutations, and the nth letter will have n positions in each of these permutations. Thus, *the permutations of* n *letters will be equal to the permutations of* n — 1 *letters multiplied by* n.

Let Q represent the permutations of $n - 1$ letters. The permutations of n letters will then be $Q\,n$. And $Q\,n$ will be the general formula for permutation.

In the general formula let $n = 2$, then Q will be 1; whence 1×2 will be the permutations of two letters.

Again, let $n = 3$, then Q will be 1×2; whence $1 \times 2 \times 3$ will be the permutations of three letters.

In like manner $1 \times 2 \times 3 \times 4$ will be the permutations of 4 letters, and so on. The following rule for permutation will, therefore, be readily inferred, viz.:

To find the permutations of a given number of things: *Multiply together the series of numbers,* 1, 2, 3, *&c., to the given number inclusive; the product will be the permutations sought.*

Ex. 1. How many changes can be made in the order of the nine digits? Ans. 362880.

Ex. 2. How often can eight persons, standing near one another, change their places, so as to present a different order each time? Ans. 40320 times.

Ex. 3. Five gentlemen agreed to board together, as long as they could seat themselves every day in a different position at dinner table. How long did they board together? Ans. 120 days.

93. In the preceding examples the things are supposed each to be different. When this is not the case, the number of changes will be less. Thus, let there be five letters, $a\,a\,b\,b\,b$. If the letters were all different, the permutations would be $1 \times 2 \times 3 \times 4 \times 5$. But since there are two a's, there will be 1×2 permutations in which the a will be combined in the same manner with the other changes. On this account, the whole number of permutations must be divided by 1×2. In like manner, on account of the three b's, the whole number of permutations must be divided by $1 \times 2 \times 3$. The number of permutations of the given letters will be, therefore,

$$\frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times 2 \times 1 \times 2 \times 3} = 10.$$

The learner will readily form a rule for this case. We shall merely subjoin a few examples.

Ex. 1. In how many ways may the figures of the number 36376 be arranged? Ans. 30.

Ex. 2. In how many ways may the letters in the word Philadelphia be arranged? Ans. 14968800.

Ex. 3. How many permutations can be made with the letters in the word Cincinnati? Ans. 50400.

SECTION X. — Square Root of Algebraic Quantities.

94. Let the proposed be a monomial. In order to find the process for extracting the root, let us observe how a monomial is raised to the square.

We will take, for example, the monomial $5\,a^2\,b^3\,c$. By the rule for multiplication, we have

$$(5\,a^2\,b^3\,c)^2 = 5\,a^2\,b^3\,c \times 5\,a^2\,b^3\,c = 25\,a^4\,b^6\,c^2.$$

A monomial is, therefore, raised to the square by squaring the coefficient and doubling the exponents of each of the letters. Hence, reversing the process, we have the following rule for the extraction of the square root of a monomial

1°. *Extract the square root of the coefficient.* 2°. *Divide the exponent of each letter by 2.*

Ex. 1. What is the square root of $64\,a^4\,b^2$?

ANS. $8\,a^2\,b$.

Ex. 2. What is the square root of $169\,a^8\,b^4\,c^2$?

ANS. $13\,a^4\,b^2\,c$.

Ex. 3. What is the square root of $144\,x^6\,y^4\,z^2$?

ANS. $12\,x^3\,y^2\,z$.

Ex. 4. What is the square root of $441\,x^{10}\,y^4\,z^6$?

ANS. $21\,x^5\,y^2\,z^3$.

Ex. 5. What is the square root of $576\,a^4\,b^8\,c^2$?

ANS. $24\,a^2\,b^4\,c$.

95. It follows, from the rule for raising monomials to the square, that a monomial, to be a perfect square, must have its coefficient a perfect square, and all its exponents even numbers.

When this is not the case, the quantity is not a perfect square. Thus $75\,a^4\,b$ is not a perfect square. Its root can, therefore, only be indicated by means of the radical sign, thus: $\sqrt{75\,a^4\,b}$.

Quantities of this kind are called *irrational* quantities of the second degree, or, more simply, *radicals* of the second degree.

96. The second power of a product, it is easy to see, is the same as the product of the second powers of all its factors. It follows, therefore, that *the square root of a product will be the same as the product of the square roots of all its factors.*

By means of this principle, radical expressions may frequently be reduced to a more simple form. Thus, the above expression, $\sqrt{75\,a^4\,b}$, may be put under the form $\sqrt{25\,a^4} \times \sqrt{3\,b}$; but $\sqrt{25\,a^4} = 5\,a^2$; whence

$$\sqrt{75\,a^4\,b} = 5\,a^2\,\sqrt{3\,b}.$$

In like manner,

$$\sqrt{98\,a\,b^4} = \sqrt{49\,b^4 \times 2\,a} = 7\,b^2\,\sqrt{2\,a}.$$

The quantities which stand without the radical sign are called the coefficients of the radical. Thus in the expressions

$$5\,a^2\,\sqrt{3\,b}, \quad 7\,b^2\,\sqrt{2\,a},$$

$5\,a^2$, $7\,b^2$ are called the coefficients of the radicals.

From what has been done, we have the following rule for reducing a radical expression of the second degree to its most simple state:

1°. *Separate the expression into two parts, one of which shall contain all the factors which are perfect squares, and the other the remaining ones.*

2°. *Take the roots of the perfect squares and place them before the radical sign, under which leave those factors which are not perfect squares.*

To determine if a number has a factor which is a perfect square, we see if it is divisible by either of the perfect squares,

4, 9, 16, 25, 36, 49, 64, 81, &c.

EXAMPLES.

1. Reduce $\sqrt{32\,a^7\,b}$ to its simplest form.

Ans. $4\,a^3\,\sqrt{2\,a\,b}$.

2. Reduce $\sqrt{128\,a^3\,b^2}$ to its simplest form.

Ans. $8\,a\,b\,\sqrt{2\,a}$.

3. Reduce $\sqrt{243\,a^9\,b^3}$ to its simplest form.

Ans. $9\,a^4\,b\,\sqrt{3\,a\,b}$.

4. Reduce $\sqrt{729\,a^7\,b^5\,c^4\,d}$ to its simplest form.

ANS. $27\,a^3\,b^2\,c^2\,\sqrt{a\,b\,d}$.

5. Reduce $\sqrt{675\,a^3\,b^7\,c^3\,d^2}$ to its simplest form.

ANS. $15\,a\,b^3\,c\,d\,\sqrt{3\,a\,b\,c}$.

97. Since the square of $-a$, as well as that of $+a$, is a^2, conversely, the root of a^2 may be either $+a$, or $-a$. Both of these roots may be comprehended in one expression by means of the double sign $\pm$. Thus,

$$\sqrt{a^2} = \pm\,a,\ \sqrt{25\,b^4\,c^2} = \pm\,5\,b^2\,c.$$

The double sign, it is evident, should be considered as affecting the square root of all quantities whatever.

If the proposed monomial be negative, the square root is impossible, since there is no quantity, positive or negative, which, multiplied by itself, will produce a negative quantity. Thus, $\sqrt{-a}$, $\sqrt{-3\,b^2}$, are *impossible* or *imaginary* quantities.

Expressions of this kind may be simplified in the same manner as radical expressions which are real. Thus, $\sqrt{-9}$ may be put under the form $\sqrt{-1 \times 9}$;

whence $$\sqrt{-9} = 3\,\sqrt{-1}.$$

In like manner $\sqrt{-4\,a^2} = 2\,a\,\sqrt{-1}$.

98. We pass to the extraction of the square root of polynomials.

1. Let the proposed be $a^2 + b^2$. This, it is evident, cannot be a perfect square; for the square of a monomial will be a monomial, and the square of a binomial consists always of three terms.

2. Let us begin, then, with a quantity consisting of three terms, the following, for example:

$$12\,a^3\,b^3 + 4\,a^2\,b^4 + 9\,a^4\,b^2.$$

The process is analogous to that which we have already pursued, in relation to the square root of numbers. The latter is, indeed, derived from the former.

In order to return from the proposed to its root, let us observe how the two terms of a binomial are employed in forming its square. Let the binomial be $m + n$, for example: then $(m + n)^2 = m^2 + 2\,m\,n + n^2$.

Thus the square of a binomial consists of three terms, viz.,

1°. *The square of the first term.*

2°. *Twice the product of the first term by the second.*

3°. *The square of the second term.*

Arranging the proposed with reference to some letter, *a*, for example, we have

$$9\,a^4\,b^2 + 12\,a^3\,b^3 + 4\,a^2\,b^4;$$

the first term of which, it is evident, is the square of the first term of the root sought. Taking the root of this, we have $3\,a^2\,b$ for the first term of the root. Dividing next the second term $12\,a^3\,b^3$ by $6\,a^2\,b$, twice the term already found, we have $2\,a\,b^2$ for the second term of the root; and since the square of this is equal to $4\,a^2\,b^4$ the remaining term of the proposed, the proposed is a perfect square, the root of which is $3\,a^2\,b + 2\,a\,b^2$.

Ex. 1. What is the square root of

$$25\,a^2\,b^4 + 30\,a^2\,b^3 + 9\,a^2\,b^2?$$

Ex. 2. What is the square root of

$$49\,a^6\,b^4 + 56\,a^5\,b^3 + 16\,a^4\,b^2?$$

Ex. 3. What is the square root of

$$81\,a^2\,b^4 - 54\,a^3\,b^3 + 9\,a^4\,b^2?$$

Ex. 4. What is the square root of

$$100\,a^6\,b^2 - 140\,a^5\,b^3 + 49\,a^4\,b^4?$$

3. Again, let the proposed consist of more than three terms, the root will consist of more than two terms. Let it consist

of three; and let us observe the manner in which the terms of a trinomial are employed in forming its square.

Let the trinomial root be $m + n + p$, for example. This may be put under a binomial form, thus, $(m + n) + p$; and, forming its square after the manner of a binomial, we have for the square,

$$(m + n)^2 + 2 (m + n) p + p^2.$$

Let the proposed, arranged with reference to the letter x, be $$9 x^4 - 12 x^3 + 16 x^2 - 8 x + 4.$$

The square of the first two terms of the root will be found in the first three terms $9 x^4 - 12 x^3 + 16 x^2$, the last of which is affected by terms united with it from the two other parts of the square. Recollecting this, and proceeding as above, we obtain $3 x^2 - 2 x$ for the first two terms of the root sought.

Subtracting next the square of $3 x^2 - 2 x$ from the proposed, there remains $12 x^2 - 8 x + 4$. This will contain twice the product of the two terms of the root already found by the third term of the root, together with the square of this third term. Dividing next the remainder by $2 (3 x^2 - 2 x)$, or, which is the same thing, dividing its first term by $6 x^2$, we obtain 2 for the third term of the root. Subtracting finally the product of $2 (3 x^2 - 2 x)$ by 2, together with the square of 2 from the remainder, $12 x^2 - 8 x + 4$, 0 remains. The proposed is, therefore, a perfect square, the root of which is

$$3 x^2 - 2 x + 2.$$

The calculations may be performed and disposed as follows:

$$
\begin{array}{l|l}
9x^4 - 12x^3 + 16x^2 - 8x + 4 & 3x^2 - 2x + 2 \\
9x^4 & \overline{6x^2 - 2x} \\
\hline
\quad - 12x^3 + 16x^2 & 6x^2 - 4x + 2 \\
\quad - 12x^3 + \ \ 4x^2 & \\
\hline
\qquad\qquad 12x^2 - 8x + 4 & \\
\qquad\qquad 12x^2 - 8x + 4 & \\
\hline
\end{array}
$$

The process, it is evident, will be the same, whatever the number of terms in the proposed. We have, therefore, the following rule for the extraction of the square root of algebraic quantities.

1°. *Arrange the proposed with reference to the powers of some letter.*

2°. *Take the square root of the first term. This will be the first term of the root. Subtract its square from the given quantity, and set down the remainder for a dividend.*

3°. *Divide the first term of the dividend by twice the root already found; the quotient will be the second term of the root, which place in the root and also in the divisor.*

4°. *Multiply the divisor thus completed, by the term of the root last found, and subtract the product from the dividend, and so proceed.*

EXAMPLES.

1. Extract the square root of

$$a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

Ans. $a^2 + 2ax + x^2$.

2. Extract the square root of

$$9x^2 - 12x + 6xy + y^2 - 4y + 4.$$

Ans. $3x + y - 2$.

3. Extract the square root of

$$x^6 + 4x^5 + 2x^4 + 9x^2 - 4x + 4.$$

Ans. $x^3 + 2x^2 - x + 2$.

4. Extract the square root of

$$9x^6 - 12x^5 + 10x^4 - 28x^3 + 17x^2 - 8x + 16.$$

ANS. $3x^3 - 2x^2 + x - 4.$

5. Extract the square root of

$$4a^4 - 20a^3x + 37a^2x^2 - 30ax^3 + 9x^4.$$

ANS. $2a^2 - 5ax + 3x^2.$

6. Extract the square root of

$$x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6.$$

ANS. $x^3 - 3ax^2 + 3a^2x - a^3.$

SECTION XI.—EQUATIONS OF THE SECOND DEGREE

99. An equation which contains the second power of the unknown quantity, without any of the higher powers, is called an equation of the *second degree.*

In an equation of the second degree, there can be, therefore, only three kinds of terms, viz.: 1°. Terms which involve the second power of the unknown quantity; 2°. Terms which involve the first power of the unknown quantity; 3°. Terms consisting entirely of known quantities.

Equations of the second degree are divided into two classes, according as they involve the first, or the first and second of these two kinds of terms, together with terms altogether known.

1°. Equations which involve the square only of the unknown quantity, and terms altogether known. These are called *Incomplete Equations.* They are sometimes also called *Pure Equations.*

2°. Equations which involve both the first and second powers of the unknown quantity, and known terms. These are called *Complete Equations;* sometimes also *Affected Equations.*

INCOMPLETE EQUATIONS.

100. We commence with incomplete equations.

1. What number is that whose half multiplied by its third is equal to 864?

Let $x =$ the number.

Then, by the question,

$$\frac{x}{2} \times \frac{x}{3} = 864;$$

whence $x^2 = 5184,$

and extracting the square root of both sides,

$$x = 72.$$

Equations of the second degree admit of two values for the unknown quantity, while those of the first degree admit of but one only. This arises from the circumstance that the second power of a quantity will be positive, whether the quantity itself be positive or negative. Thus, in the preceding example, x will be equal either to $+ 72$, or $- 72$; or, uniting both values in one expression,

$$x = \pm 72.$$

2. What two numbers are those whose sum is to the greater as 10 to 7, and whose sum multiplied by the less produces 270?

Let $x =$ their sum.

Then $\frac{7x}{10}$ will be the greater, and $\frac{3x}{10}$ the less:

and, by the question, $\frac{3x^2}{10} = 270;$

whence $x^2 = 900,$

and $x = \pm 30.$

The numbers will be, therefore, ± 21, and ± 9.

3. There are two numbers in the proportion of 4 to 5 the difference of whose squares is 81. What are those numbers?

Let $x =$ the greater; then $\frac{4x}{5} =$ the less, and we have

$$x^2 - \frac{16x^2}{25} = 81;$$

whence $x^2 = 225,$

and $x = \pm 15.$

The numbers, therefore, are 15 and 12.

4. It is required to find the value of x in the equation,

$$\frac{5}{7}x^2 - 8 = 4 - \frac{2}{3}x^2.$$

Freeing from denominators, transposing and reducing,

$$x^2 = \frac{252}{29}; \text{ whence } x = \pm\sqrt{\frac{252}{29}}.$$

In this example $\frac{252}{29}$ is not a perfect square; we can therefore obtain only an approximate value for x.

5. What is the value of x in the equation $x^2 + 25 = 9$?

Transposing, we have $x^2 = -16;$

whence $x = \pm\sqrt{-16}.$

To find the value of x, we are here required to extract the square root of -16. But this is impossible; for, since there is no quantity, positive or negative, which multiplied by itself will produce a negative quantity, -16, it is evident, cannot have a square root either *exact* or *approximate*. Indeed, -16 may be considered as arising from the multiplication of $+4$ by -4; but $+4$ and -4 are different quantities; their product, therefore, is not a square.

The result $x = \sqrt{-16}$ shows, then, that it is impossible to resolve the equation from which it is derived. It implies, therefore, that there is some absurdity or impossibility in the conditions of the question which has led to it.

In general, an expression of the square root of a negative quantity is to be regarded as a symbol of *impossibility.*

101. Incomplete equations of the second degree may always be reduced to an equation of the form

$$A x^2 = B;$$

A and B denoting any known quantities whatever, positive or negative.

This may be done, 1°. By collecting into one member the terms which involve x^2, and reducing them to one term; and, 2°. By collecting the known terms into the other member.

Ex. 1. Reduce the equation $\frac{x^2}{3} - 4 = \frac{x^2}{7} + 9$ to the form $A x^2 = B$.

Ans. $4 x^2 = 273$; in which $4 = A$, $273 = B$.

Ex. 2. Reduce the equation $x^2 - 192 = \frac{x^2}{4}$ to the form $A x^2 = B$.

Ex. 3. Reduce the equation $\frac{x^2}{3} - 8 = \frac{x^2}{9} + 10$ to the form $A x^2 = B$.

Ex. 4. Reduce the equation $a x^2 - b = c x^2 - d$ to the form $A x^2 = B$.

Ans. $(a - c) x^2 = b - d$;

in which $a - c = A$, $b - d = B$.

Resolving, next, the equation $A x^2 = B$, we obtain

$$x = \pm \sqrt{\frac{B}{A}}.$$

If $\frac{B}{A}$ is a perfect square, the value of x may be obtained exactly; if not, it may be found with such degree of approximation as we please. If $\frac{B}{A}$ be negative, we shall have $\sqrt{-\frac{B}{A}}$, a symbol of impossibility.

102. From what has been done, we have the following rule for the solution of incomplete equations of the second degree.

1°. *Reduce the equation to the form* $Ax^2 = B$. 2°. *Divide both sides by the coefficient of* x^2, *and then extract the square root of both members.*

EXAMPLES.

1. What number is that which being multiplied by itself, the product will be 529? Ans. 23.

2. A gentleman being asked how much money he had, replied: "If to three times the square of the dollars you add $8, and from five times the square you subtract $10, the sum and difference will be equal." How much money had he? Ans. $3.

3. Find three numbers in the proportion of 2, 3, and 5, the sum of whose squares is 342.

Ans. 6, 9, and 15.

4. What two numbers are those which are to each other as 3 to 4, and the difference of whose squares is 28?

Ans. 6 and 8.

5. What number is that whose 7th and 8th parts multiplied together, and the product divided by 3, gives the quotient $298\frac{2}{3}$? Ans. 224.

6. It is required to find a number such that if we first add to it 94, and then subtract it from 94, the product of the sum and difference thus obtained will be 8512. Ans. 18.

7. A man lent a certain sum of money at 6 per cent. a year, and found that if he multiplied the principal by the number representing the interest for 8 months, the product would be $900. Required the principal. ANS. $150.

8. A gentleman has two square parlors, the sides of which are as 2 to 3; and he finds that it takes 180 square feet of carpeting more to cover the floor of the larger than it does to cover that of the smaller. What is the length of one side of each room?

ANS. 18 and 12 feet respectively.

9. The sides of a rectangular field are to each other as 5 to 7, and there are 140 square rods in the field. What are the lengths of its sides? ANS. 10 and 14 rods.

10. A says to B, My son's age is one quarter of yours, and the difference between the squares of the numbers representing their ages is 240. What are their ages?

ANS. 16 and 4 years.

11. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men, and by this movement the detachment was drawn up in five lines. Required the number of men.

ANS. 4550.

12. What number is that, to the square of which if 50 be added, the sum will be 40?

ANS. The question implies an impossibility.

103. With a little care, equations may be solved by the present rule, which otherwise would require that for the second class of equations of the second degree.

We give a few examples.

1. It is required to divide the number 14 into two such parts that the quotient of the greater part divided by the

less may be to the quotient of the less divided by the greater as 16 to 9.

Let $x =$ the greater part; then $14 - x =$ the less; and, by the question,

$$\frac{x}{14 - x} : \frac{14 - x}{x} :: 16 : 9;$$

whence
$$9 \frac{x}{14 - x} = 16 \frac{14 - x}{x}. \quad (1)$$

Freeing from denominators and reducing, taking care to retain the operations as we proceed, we obtain

$$9 x^2 = 16 (14 - x)^2, \quad (2)$$

and
$$\frac{x^2}{(14 - x)^2} = \frac{16}{9}. \quad (3)$$

Extracting next the square root of both sides,

$$\frac{x}{14 - x} = \frac{4}{3};$$

whence
$$x = 8.$$

We have deduced the equation (3) from (2) in order, in accordance with the general rule for incomplete equations, to bring, before extracting the root, the terms altogether known upon one side of the equation, and those which involve x upon the other.

Recollecting, however, that the square root of a product is equal to the product of the square root of each of its factors, equation (2) gives, by extracting the root of both sides,
$$3 x = 4 (14 - x),$$
which gives, as before, $x = 8$.

2. Two travellers, A and B, set out to meet each other, A leaving the town C at the same time that B left D. They travelled the direct road C D, and, on meeting, it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have

been 28 days in performing A's journey. What was the distance between C and D?

Let $x =$ the number of miles A has travelled; then

$x - 18 =$ the number B has travelled; and

$$\frac{15\frac{3}{4}}{x-18} = \text{the time A would travel 1 mile,}$$

$$\frac{28}{x} = \text{the time B would travel 1 mile;}$$

whence, as they are each the same time upon the road,

$$\frac{63\,x}{4\,(x-18)} = \frac{28\,(x-18)}{x};$$

from which we obtain

$$63\,x^2 = 112\,(x-18)^2.$$

Here the coefficients, 63 and 112, it is evident, are not perfect squares. But, with a little attention, it will be seen that they each contain a factor, 7, which being struck out will leave them perfect squares. Dividing, therefore, both sides by 7, we have

$$9\,x^2 = 16\,(x-18)^2;$$

whence $3\,x = 4\,(x-18)$,

and $x = 72$;

whence A travelled 72, and B 54 miles; and the whole distance C D is 126 miles.

3. Divide the number 49 into two such parts that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

Ans. 28 and 21.

4. Divide the number 24 into two parts whose squares shall be as 25 to 9. Ans. 15 and 9.

5. A farmer bought two flocks of sheep, the first of which contained 18 fewer than the second. If he had given for the first flock as many pounds as there were sheep in the

second, and for the second as many pounds as there were sheep in the first, then the price of 6 sheep of the first flock would have been to the price of 7 sheep of the second as 7 to 6. Required the number in each flock.

ANS. 108 and 126.

6. Two men share a prize of \$40 in such a manner that the quotient of A's share divided by twice B's share is to the quotient of B's share divided by three times A's as 54 to 9. What is the share of each?

ANS. A's \$26$\frac{2}{3}$, B's \$13$\frac{1}{3}$.

7. Bought a number of oxen for \$1406.25, the number of dollars per head being to the number of oxen as 9 to 4. How many did I buy, and what did I give for each?

ANS. 25 oxen, at \$56.25 per head.

8. A merchant purchased two pieces of cloth, which together measured 100 yards. The number of yards in the first, divided by the number of yards in the second, was to the number of yards in the second, divided by 5 times the number of yards in the first, as 45 to 25. How many yards were there in each piece? ANS. 37$\frac{1}{2}$ and 62$\frac{1}{2}$ yards.

104. In the preceding examples, we have employed one unknown quantity only. If there are two unknown quantities employed, we eliminate one of them, and then apply the rule to the new equation thus obtained.

The following examples will serve as an exercise upon questions involving naturally two unknown quantities:

1. The sum of two numbers is to the greater as 7 to 5; and if the sum be multiplied by the less, the product will be 126. What are the numbers?

Let $x =$ the greater number, and y the less.

By the question,

$$x + y : x :: 7 : 5, \quad (1)$$

and $$xy + y^2 = 126. \quad (2)$$

The value of x derived from (1) and substituted in (2) gives $$\frac{5}{2}y^2 + y^2 = 126;$$

from which we obtain $y = 6$;

whence $x = 15$.

2. The product of two numbers is 63, and the square of their sum is to the square of their difference, as 64 to 1. What are the numbers?

By the first condition,

$$xy = 63.$$

By the second,

$$(x + y)^2 = 64(x - y)^2.$$

Extracting the root of both sides of this last,

$$x + y = 8(x - y).$$

From which, if we deduce the value of x, and substitute it in the first equation, we obtain,

$$y^2 = 49,$$

and the numbers will be 7 and 9.

3. There are two numbers in the proportion of 8 to 5, whose product is 360. What are the numbers?

Ans. 24 and 15.

4. There is a room of such dimensions that the difference of the sides, multiplied by the less, is equal to 36, and the product of the sides is equal to 360. What are the sides?

Ans. 18 and 20.

5. The sides of a rectangular field are to each other as 5 to 7, and its area is 26 *A.* 1 *r.* 35 *p.* How many rods are there in each side? Ans. 55 and 77.

6. A trader sold two pieces of broadcloth, which together measured 18 yards; and he received as many dollars a yard for each piece as it contained yards. Now the sums re-

ceived for the two were to each other as 25 to 16. How many yards were there in each piece?

ANS. 10 and 8 yards.

7. A merchant purchased two pieces of muslin, which together measured 12 yards. He gave for each piece just as many dollars per yard as the piece contained yards; and he gave 4 times as much for one piece as for the other. How many yards were there in each piece?

ANS. 8 and 4.

8. A certain room requires 108 square feet of carpeting to cover it; and the sum of its length and breadth is equal to twice their difference. What is the length and breadth of the room? ANS. 18 and 6 feet respectively.

9. A charitable person distributed a certain sum among some poor men and women, the numbers of which were in the proportion of 4 to 5. Each man received one-third as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. The men received, altogether, 18 shillings more than the women. How many were there of each? ANS. 12 men and 15 women.

10. The sum of the squares of two numbers is 117, and the difference of their squares is 45; what are the numbers?

ANS. 9 and 6.

11. There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers? ANS. 18 and 14.

12. A sum of 152 dollars is divided among a certain number of men and boys. The number of men is to the number of boys as 3 to 4. Now the boys receive each one-half as many dollars as there are persons, and the men twice

as many dollars each as there are boys. How many men are there, and how many boys? ANS. 6 men and 8 boys.

13. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number?

ANS. 23.

14. From two towns, C and D, which were at the distance of 396 miles, two persons, A and B, set out at the same time, and meet each other, after travelling as many days as are equal to the difference of the number of miles they travelled per day; when it appears that A has travelled 216 miles. How many miles did each travel per day?

ANS. A 36, and B 30.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

105. We pass, next, to the solution of complete equations of the second degree. Let us take, for example, the equation

$$x^2 + 8x = 209.$$

The solution of this equation, it is evident, would present no difficulty if the left hand member were a perfect square. But this is not the case; for the square of a quantity consisting of one term will consist of one term, and the square of a binomial contains three terms.

Let us see, then, if $x^2 + 8x$ can be made a perfect square. A binomial $x + a$ raised to the square, gives, as we have seen,

$$(x + a)^2 = x^2 + 2ax + a^2.$$

If, then, we compare $x^2 + 8x$ with $x^2 + 2ax + a^2$, it is evident that $x^2 + 8x$ may be considered the first and second terms in the square of a binomial. The first term of this binomial will evidently be x; then, as $8x$ must contain twice the first term by the second, the second will be found by dividing $8x$ by $2x$, which gives 4 for the quotient. $x^2 + 8x$, form, therefore, the first two terms in the square of the binomial $x + 4$. If, then, we add 16, the square of 4, to $x^2 + 8x$, the left-hand member of the proposed, the result $x^2 + 8x + 16$ will be a perfect square. But if 16 be added to the left-hand member, it must also be added to the right, in order to preserve the equality; the proposed will then become

$$x^2 + 8x + 16 = 225,$$

whence, extracting the root of each member,

$$x + 4 = \pm 15;$$

wherefore, $\quad x = 11, x = -19.$

2. Let us take, as a second example, the equation

$$x^2 - \frac{2}{3}x = 15\tfrac{8}{9}.$$

Comparing $x^2 - \frac{2}{3}x$ with the square of the binomial $x - a$, viz., $x^2 - 2ax + a^2$, it is evident that $x^2 - \frac{2}{3}x$ may be considered the first two terms in the square of a binomial. Pursuing the same course of reasoning as in the preceding example, we find this binomial to be $x - \frac{1}{3}$. If, then, the square of $\frac{1}{3}$ be added to both sides, the left-hand member will be a perfect square, and we have

$$x^2 - \frac{2}{3}x + \frac{1}{9} = 16.$$

Extracting, next, the root of each member,

$$x - \frac{1}{3} = \pm 4;$$

whence $\quad x = 4\tfrac{1}{3}, x = -3\tfrac{2}{3}.$

Making the left-hand member a perfect square, is called *completing the square.* This is done, in the preceding examples, *by adding to both sides of the equation the square of one-half the coefficient of* x *in the second term.*

106. Complete equations of the second degree may always be reduced to the form $x^2 + p x = q$; p and q denoting any quantities whatever, positive or negative, entire or fractional.

This is done, 1°. By collecting all the terms which involve x into one member, and the terms altogether known into the other. 2°. By uniting the terms which contain x^2 into one term, and those which contain x into another. 3°. By changing the signs of each term, if necessary, in order to render that containing x^2 positive. 4°. By dividing all the terms by the multiplier of x^2, if it have a multiplier, and multiplying all the terms by the divisor of x^2, if it have a divisor.

Thus, let there be the equation,

$$7 - \frac{3}{5} x = \frac{61 - x^2}{4 x - 2}.$$

Freeing from denominators,

$$140 x - 70 - 12 x^2 + 6 x = 305 - 5 x^2.$$

Transposing and uniting terms,

$$146 x - 7 x^2 = 375.$$

Changing the signs of each term,

$$7 x^2 - 146 x = - 375.$$

Dividing by 7, the coefficient of x,

$$x^2 - \frac{146 x}{7} = - \frac{375}{7}.$$

Comparing this last with the equation $x^2 + p x = q$, its form, it is evident, is the same; and we have

$$p = -\frac{146}{7},\ q = -\frac{375}{7}.$$

This being premised, $x^2 + p\,x$ form, it is evident, the first two terms of the square of the binomial,

$$x + \frac{p}{2};$$

since we have $\left(x + \frac{p}{2}\right)^2 = x^2 + p\,x + \frac{p^2}{4}.$

Adding, then, $\frac{p^2}{4}$, the square of half the coefficient of the second term in the first member, to both sides of the equation, $x^2 + p\,x = q$, we have

$$x^2 + p x + \frac{p^2}{4} = q + \frac{p^2}{4}.$$

Extracting the root of both sides,

$$x + \frac{p}{2} = \pm\sqrt{q + \frac{p^2}{4}};\qquad \text{whence}$$

$$x = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}},\ x = -\frac{p}{2} - \sqrt{q + \frac{p}{4}}$$

This is a general solution of complete equations of the second degree. We have, therefore, the following rule for the solution of equations of this kind, viz.: 1°. *Reduce the equation to the form* $x^2 + p\,x = q$. 2°. *Complete the square by adding to both members the square of half the coefficient of* x *in the second term.* 3°. *Extract the root of both members, taking care to give to the root of the second member the double sign.* 4°. *Deduce the value of* x *from the equation which results from the last operation.*

EXAMPLES.

1. Given $x^2 + 6\,x = 55$, to find the values of x.

Ans. $x = 5$, or -11.

2. Given $10x^2 - 8x + 6 = 318$, to find the values of x. ANS. $x = 6$, or $-5\frac{1}{5}$.

3. Given $x^2 - 10x + 15 = \frac{x^2}{5} - 34x + 155$, to find the values of x. ANS. $x = 5$, or -35.

4. Given $2x^2 + 8x + 7 = \frac{5x}{4} - \frac{x^2}{8} + 197$, to find the values of x. ANS. $x = 8$, or $-11\frac{3}{17}$.

5. Given $\frac{x^2}{4} - \frac{x}{3} + 15 = \frac{x^2}{9} - 8x + 95\frac{1}{4}$, to find the values of x. ANS. $x = 9$, or $-64\frac{1}{5}$.

6. Given $x + 4 + \frac{7x - 8}{x} - 5 = 8$, to find the values of x. ANS. 4, or -2.

7. Given $3x - \frac{1121 - 4x}{x} = 2$, to find the values of x. ANS. $x = 19$, or $-19\frac{2}{3}$.

8. Given $5x - \frac{3x - 3}{x - 3} = 2x + \frac{3x - 6}{2}$, to find the values of x. ANS. $x = 4$, or -1.

107. We pass to the solution of some questions, confining ourselves, for the present, to the value obtained for x when the positive value only of the quantity under the radical is employed.

1. A gentleman divided \$28 between his two sons in such a manner that the product of their shares was 192. What was the share of each? ANS. \$16 and \$12.

2. There are two square courts that are paved with stones, a foot square each. The side of one court exceeds the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately?
ANS. 26 and 38 feet

3. A laborer dug two trenches, one of which was 6 yards longer than the other, for £17 16*s.*; and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

ANS. 10 and 16 yards.

4. A person buys some pieces of cloth, at equal prices, for $60. Had he got 3 more pieces for the same sum, each piece would have cost him 1 dollar less. How many pieces did he buy? ANS. 12.

5. There are two numbers, one of which is 8 more than the other, and their product is 240. Required the numbers.

ANS. 12 and 20.

6. A merchant sold a piece of cloth for $39, and gained as much per cent. as it cost him. What was the cost of the cloth? ANS. $30.

7. The length of a room exceeds its width by 8 feet, and its area is 768 feet. What is its length and width?

ANS. 32 and 24 feet.

8. A man, having travelled 160 miles, found that if he had travelled one mile more per hour, he would have been 8 hours less upon the road. Required his rate of travelling, and the number of hours he was upon the road.

ANS. 4 miles an hour, and 40 hours.

9. A man has a painting 18 inches long and 12 inches wide, which he orders the cabinet-maker to put into a frame of uniform width, and to have the area of the frame equal to that of the painting. Of what width will the frame be?

ANS. 3 inches.

10. A person bought some yards of cloth for 120 shillings; if there had been 6 yards more, each yard would have cost a shilling less. Required the number of yards and the price of each. ANS. 24 yards, at 5*s.* a yard.

11. Two men, A and B, traded together. A put in a certain sum for 4 months, and B put in $350 for 2 months. They gained $99, and A received for principal and gain $136. How much stock did A put in? Ans. $100.

12. Several gentlemen made a purchase in company for 175 dollars. Two of them having withdrawn, the bill was paid by the others, each furnishing 10 dollars more than would have been his equal share if the bill had been paid by the whole company. What was the number in the company at first? Ans. 7.

13. A merchant bought several yards of linen for 60 dollars, out of which he reserved 30 yards, and sold the remainder for 54 dollars, gaining 10 cents a yard. How many yards did he buy, and at what price?

Ans. 120 yards, at 50 cents.

14. Two casks of wine were purchased for $58, one of which contained 5 gallons more than the other, and the price by the gallon was $2 less than one-third of the number of gallons in the smaller cask. Required the number of gallons in each, and the price by the gallon.

Ans. 12 and 17 gallons, at $2.

15. A gentleman purchased a building lot, and in the centre of it erected a house 54 feet long and 36 feet wide, which covered just one-half of his land. This arrangement left him a flower border of uniform width all round his house. What was the width of his border, and the dimensions of his lot?

Ans. The border 9 feet, and the lot 72 by 54 feet.

16. A father left an estate of $30000 to be divided equally among his sons; but one of these dying immediately after his father, the estate was divided among those remaining, each of whom received $1500 more than he

would have received if all had been living. How many sons did the father leave? Ans. 5.

17. A charitable person divides the sum of $36, in equal shares, among the poor of a small town; but as 6 of those whom he thought of relieving stood no longer in need of assistance, each of the remaining paupers had for his share one-twelfth of a dollar more than he otherwise would have had. How many paupers were there at first? Ans. 54.

NEGATIVE VALUES.

108. Thus far we have considered the positive values only obtained for the unknown quantity. Let us now examine, in some examples, the negative values also.

1. A person bought some sheep for $72; and found, if he had bought 6 more for the same money, he would have paid $1 less for each. How many did he buy?

Let x equal the number; then, by the question,

$$\frac{72}{x} - \frac{72}{x+6} = 1;$$

from which we obtain $x = 18$, $x = -24$.

The positive value of x satisfies the question in the exact sense of the enunciation. To interpret the negative result, we substitute $-x$ for x in the equation, which becomes

$$\frac{72}{-x} - \frac{72}{-x+6} = 1;$$

or, which is the same thing,

$$\frac{72}{x-6} - \frac{72}{x} = 1,$$

an equation which corresponds to the following enunciation.

A person bought some sheep for $72; and found, if he had bought 6 *less* for the same money, he would have paid $1 *more* for each. How many did he buy?

The negative value here modifies the question in a manner analogous to what takes place, as we have already seen, in equations of the first degree.

2. Bought some cloth for \$24, for which I paid \$2 more per yard than there were yards in length. How many yards were there? ANS. 4, or — 6.

How shall we interpret the negative answer to this question?

3. A company at a tavern had £8 15*s*. to pay; but two of them having left before the bill was paid, those who remained had each 10 shillings more to pay. How many were in the company at first? ANS. 7, or — 5.

In what way must this question be modified, so that the negative value may become positive and answer its conditions?

4. To find a number such, that if the square of this number be augmented by 5 times the number and also by 6, the result will be 2.

Putting x for the answer, the values of x in this case will both be negative. The question, therefore, is impossible in the sense in which it is enunciated. How must it be modified, so as to become capable of solution?

5. To divide the number 10 into two such parts, that the product of these parts shall be 30.

Putting x for one of the parts, $10 - x$ will be the other. We have, therefore, by the question,

$$10x - x^2 = 30.$$

Resolving this equation, we obtain

$$x = 5 + \sqrt{-5}, x = 5 - \sqrt{-5}.$$

This result indicates that there is some absurdity in the conditions of the question proposed; since, in order to obtain the value of x, we are required to extract the square root of a negative quantity, which is impossible.

In order to see in what this absurdity consists, let us examine into what two parts a given number should be divided, in order that the product of these parts may be the greatest possible.

Let n represent the given number, d the difference of the two parts; the greater part will then be

$$\frac{n}{2}+\frac{d}{2}, \text{ and the less, } \frac{n}{2}-\frac{d}{2}.$$

Let P represent the product of the two parts; then

$$\left(\frac{n}{2}+\frac{d}{2}\right)\left(\frac{n}{2}-\frac{d}{2}\right)=P,$$

or
$$\frac{n^2}{4}-\frac{d^2}{4}=P.$$

Here the value of P, it is evident, increases as that of d decreases. P, therefore, will be the greatest possible, when d is the least possible, or zero.

If, then, *a given number be divided into two parts, and these parts be multiplied together, the product will be the greatest possible when the parts are equal.*

Returning to the question, the greatest possible product which can be obtained by dividing 10 into two parts, and taking their product, will be 25. The absurdity of the question consists, therefore, in requiring that the product of the two parts into which 10 is to be divided should be greater than 25.

The values, $x=5+\sqrt{-5}$, $x=5-\sqrt{-5}$, in which the root of a negative quantity is required to be taken, are said to be *imaginary*, in distinction from those which can be found either exactly or by approximation, which on this account are called *real*.

GENERAL EQUATION OF THE SECOND DEGREE.

109. The equation $x^2 + px = q$ represents generally, as we have seen, any equation of the second degree, p and q being any known quantities whatever, positive or negative.

This equation, being resolved, gives

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}};$$

or, writing the values of x separately,

$$x = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}$$

$$x = -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}.$$

Any quantity which, substituted for the unknown quantity in an equation of the second degree, will satisfy it, is called a *root* of the equation.

Thus the values of x found above, are roots of the equation $x^2 + px = q$, since, if substituted for x, they will satisfy the equation. Every equation of the second degree will have, therefore, two roots, and it may be shown that they will have but two only.

If the two values of x above be added together, the sum, it is evident, will be $-p$; and if they be multiplied together, the product will be $-q$.

The roots of an equation of the second degree will have, therefore, the following properties:

1°. *The algebraic sum of the two roots is equal to the coefficient of* x *in the second term, taken with a contrary sign.*

2°. *The product of the roots is equal to the known term in the second member, taken with a contrary sign.*

Thus the roots of the equation $x^2 - 6x = 16$ are $+8$ and -2. Their sum is $+6$, and their product is -16.

Ex. 1. Find the roots of the equation

$$x^2 + 8x = 33,$$

and verify the above properties by them.

Ex. 2. Find the roots of the equation

$$x^2 - x = 42,$$

and verify the above properties by them.

Ex. 3. Find the roots of the equation

$$x^2 - 20x = -36,$$

and verify the above properties by them.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND DEGREE.

110. The solution of the equation $x^2 + px = q$ obtained above, is a general solution of equations of the second degree. We will now examine the results at which we arrive, for the various hypotheses which may be made with respect to the quantities, p and q. This is called a *discussion* of the equation.

The equation $x^2 + px = q$ will take four different forms, according to the variations of signs which may be given to p and q. These, it is evident, will be as follows, viz.:

$x^2 + px = + q$, 1st form.
$x^2 - px = + q$, 2nd form.
$x^2 + px = - q$, 3d form.
$x^2 - px = - q$, 4th form.

We will examine these in order.

First Form.

111. The first form is $x^2 + px = + q$, which, being resolved, gives

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

To obtain the value of x, we must be able to extract the root of the quantity under the radical sign, either exactly, or by approximation. This, it is evident, can always be done, since this quantity, in the present form, is necessarily positive.

The value of the radical, moreover, is greater than that of the part without; for the root of $\frac{p^2}{4}$ alone being equal to $\frac{p}{2}$, the root of $q + \frac{p^2}{4}$ will necessarily be greater than $\frac{p}{2}$. *The two roots will then be one positive and the other negative; and of these the negative root will be numerically the greater.*

Both values of x will satisfy the equation. The positive value will satisfy the question of which the equation may be considered the algebraic translation, in the exact sense of its enunciation. The negative value will require, in order that it may become an exact or positive answer, that the question should be modified in the sense of some of its conditions.

Let us illustrate by a particular case, for example, the equation $$x^2 + 8x = 20.$$

The roots of this equation are $x = 2, x = -10$. Of these, the negative root is evidently the greater.

Substituting the positive root, the equation becomes

$$2^2 + 8 \times 2 = 20; \text{ or, } 4 + 16 = 20.$$

Substituting next the negative root, it becomes

$$(-10)^2 + 8 \times -10 = 20; \text{ or, } 100 - 80 = 20.$$

Both values, therefore, satisfy the equation:

The equation may be considered as the algebraic translation of the following question:

To find a number such that if eight times the number be added to its square, the sum will be 20.

The positive value of x satisfies this question in the exact or ordinary sense of its terms.

To find the corresponding question for the negative value, we write $-x$ for x in the equation, which becomes

$$x^2 - 8x = 20,$$

and which corresponds to the following enunciation:

To find a number such that if eight times the number be *subtracted* from its square, the *remainder* will be 20.

Ex. 1. What are the roots of the equation $x^2 + 4x = 60$? How does the negative root compare numerically with the positive? What enunciation will correspond with the positive, what with the negative root?

Ex. 2. What are the roots of the equation $x^2 + 5x = 84$?

Ex. 3. What are the roots of the equation $x^2 + 4x = 140$?

Second Form.

112. The second form is $x^2 - px = +q$. This, being resolved, gives

$$x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

In this form, also, the quantity under the radical sign is necessarily positive. The value of x is, therefore, real, since the root may be taken either exactly or by approximation.

The value of the radical will also be greater than that of the other part. *The two roots, therefore, will be one positive and the other negative.* But *the positive root in this form will be numerically the greater.*

Both values of x will satisfy the equation. The positive value will satisfy the question of which the equation is the algebraic translation in the exact sense of its enunciation. The negative value corresponds to an analogous question, differing from this only in the sense of some of its conditions

Let us take, for example, the particular equation

$$x^2 - 4x = 21.$$

The roots of this equation are $+7$ and -3, of which the positive root is the greater.

Substituting first the positive root, the equation becomes

$$(7)^2 - 4 \times 7 = 21; \text{ or, } 49 - 28 = 21.$$

Substituting next the negative root, it becomes

$$(-3)^2 - 4 \times -3 = 21; \text{ or, } 9 + 12 = 21.$$

Both values, therefore, satisfy the equation. The equation may be considered as the algebraic translation of the following enunciation:

To find a number such that if four times the number be subtracted from its square, the remainder will be 21.

The positive value answers this question in its exact sense. The negative value requires it to be modified thus:

To find a number such that if four times the number be *added* to its square, the *sum* will be 21.

Each of the two preceding forms involves the other in its solution. Each, therefore, connects in itself two questions, which differ from each other only in the sense of certain conditions.

Ex. 1. What are the roots of the equation $x^2 - 6x = 16$?

Ex. 2. What are the roots of the equation $x^2 - 8x = 65$?

Third Form.

113. The third form is $x^2 + px = -q$. This, being solved, gives

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

Here, in order that the root may be extracted, or, in other words, that the value of x may be real, q must not exceed $\frac{p^2}{4}$. This being the case, the value of the radical will be less, it is evident, than that of the part without. *Both roots* *will, therefore, be negative.* These will satisfy the equation, but not the question of which it is the algebraic translation.

Let us take, as a particular example, the equation

$$x^2 + 11x = -30.$$

The roots of this equation are -6 and -5.

Substituting the first in the equation, it becomes

$$(-6)^2 + 11 \times -6 = -30;$$

or, $$36 - 66 = -30.$$

Substituting the second, it becomes

$$(-5)^2 + 11 \times -5 = -30;$$

or, $$25 - 55 = -30.$$

Both values, therefore, satisfy the equation.

Transposing the 30, the equation becomes

$$x^2 + 11x + 30 = 0,$$

and may be considered as the algebraic translation of the following enunciation:

To find a number such that if 11 times the number be added to its square, and 30 also be added to the sum, the result will be 0.

The question, as is evident from inspection, cannot be solved in the sense of its enunciation. If, however, we write $-x$ for x in the equation, it becomes

$$x^2 - 11x + 30 = 0,$$

which corresponds to the following enunciation:

To find a number such that if 11 times the number be *subtracted* from its square, and 30 be added to the *remainder* the result will be 0.

Ex. 1. What are the roots of the equation

$$x^2 + 12x = -35?$$

Ex. 2. What are the roots of the equation

$$x^2 + 9x = -20?$$

Fourth Form.

114. The fourth form is $x^2 - px = -q$. This, being resolved, gives

$$x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

Here also, in order that the value of x may be real, q must not exceed $\frac{p^2}{4}$. This condition being fulfilled, *the two roots will both be positive*, and will satisfy both the equation and the question of which it is the algebraic translation.

As a particular case, let us take the equation

$$x^2 - 5x = -6.$$

The roots of this are $+2$ and $+3$, both of which satisfy the equation. The equation returns to $5x - x^2 = 6$, which corresponds to the following enunciation:

To find a number such that 5 times the number shall exceed its square by 6.

Both the values of x satisfy this enunciation.

Ex. 1. What is the value of x in the equation

$$x^2 - 9x = -20?$$

Ex. 2. What is the value of x in the equation

$$x^2 - 11x = -12?$$

115. In the two last forms, we have seen that in order that x may be real, q must not exceed $\frac{p^2}{4}$. If q exceeds $\frac{p^2}{4}$, the value of x is imaginary, and the result indicates some absurdity in the conditions of the question of which it is the algebraic translation.

Putting $-x$ for x, the equations of the third and fourth forms return to

$$x(p - x) = q,$$

which corresponds to the following enunciation:

To divide a number p into two such parts that the product of these parts may be q.

We have seen that the product q, is the greatest possible when the two parts are equal.

By inspection of the two last formulas, moreover, it will be seen that the values of x become equal when q is equal to $\frac{p^2}{4}$. If, then, the question requires that q should be greater than this, it is manifestly impossible to fulfil its conditions, and this impossibility becomes apparent in the final result, in which we are required to extract the square root of a negative quantity.

In all cases, therefore, *when the values of the unknown quantity are imaginary, the conditions of the question are incompatible with each other.*

It will be seen, moreover, that the condition q less than $\frac{p^2}{4}$ being fulfilled, the question in the last form will admit of two direct solutions; for, the question being merely to divide a number into two parts the product of which shall be equal to a given number, there is no reason why x should represent one of the parts rather than the other; the equation, when solved, therefore, should give both at the same time.

116. The following questions will serve as an additional exercise for the learner in each of the preceding forms:

1. A draper bought a quantity of cloth for £27. If he had bought 3 yards less for the same sum, it would have cost him 15 shillings a yard more. How many yards did he buy? Ans. 12, or -9.

What modification is required in this question, in order that 9 may be the true answer?

2. To find a number such that if 180 be divided by this number, and also by this number diminished by 6, the difference of the quotients will be 5. ANS. 18, or — 12.

To what question does the negative answer belong?

3. Find a number which, subtracted from its square, makes 42. ANS. 7, or — 6.

To what question does the negative answer belong?

4. A man bought a horse, which he sold, after some time, for 24 dollars. At this sale, he loses as much per cent. upon the price of his purchase as the horse cost him. What did he pay for the horse? ANS. 60, or 40 dollars.

5. In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins; and the whole is worth 18*s*. How many are there of each?

ANS. 6 silver and 18 copper, or 18 silver and 6 copper.

6. Find a number such that if the square of this number be augmented by 5 times the number, and also by 6, the result will be 2.

How must this question be modified so that the negative answers may become positive?

7. Divide the number 44 into two such parts that their product shall be 500.

What does the imaginary value, obtained for the answer to this question, indicate? What absurdity is there in the conditions of the question?

117. We pass next to the solution of some questions involving two unknown quantities.

1. There is a number which, being divided by the product of its two digits, the quotient is 2, and if 27 be added to it, the digits will be inverted. What is the number?

Let x and y be the digits; then $10x + y =$ the number.

From the first condition,

$$\frac{10x + y}{xy} = 2, \text{ or } 10x + y = 2xy \quad (1).$$

From the second condition,

$$10x + y + 27 = 10y + x;$$

or, $$x + 3 = y \quad (2).$$

Substituting next the value of y in (1), and resolving the equation which results, we obtain $x = 3$, $y = 6$. The number is, therefore, 36.

2. The sum of two numbers is 24, and the sum of their squares is 306. What are the numbers?

ANS. 9 and 15.

3. The difference between two numbers is 15, and the square of the less is equal to one-half the greater. What are the numbers? ANS. 18 and 3.

4. The sum of two numbers is 10, and their difference divided by their product is equal to $\frac{3}{8}$. What are the numbers? ANS. 8 and 2.

5. A grocer sold 80 lbs. of mace and 100 lbs. of cloves for £65; but he sold 60 lbs. more of cloves for £20 than he did of mace for £10. What was the price of a pound of each? ANS. Mace 10*s*., cloves 5*s*.

6. A rectangular field contains 60 square rods, and the sum of its sides is to their difference as 17 to 7. What are the lengths of the sides? ANS. 12 and 5 rods.

7. There is a number consisting of two digits, that in the unit's place being the greater. The difference between the digits is 2, and if the number be divided by the product of its digits, the quotient will be 3. What is the number?

ANS. 24.

8. The sum of two numbers is to their difference as 13 to 5, and if the greater be added to the square of the less, the sum will be 25. What are the numbers?

Ans. 9 and 4.

118. The square root of a quantity, we have seen, is found by dividing its exponent by 2. Thus the square root of a^6 is $a^{\frac{6}{2}}$, or a^3. In like manner the square root of a^3 is $a^{\frac{3}{2}}$, and of a, or a^1, is $a^{\frac{1}{2}}$. The square root, then, may be indicated by the fractional exponent $\frac{1}{2}$. And, in general, the nth root of a quantity may be indicated by the fraction $\frac{1}{n}$. This last mode of notation is, in general, most convenient.

The preceding method is applicable not only to equations of the second degree, but to all equations of the form

$$x^{2n} + p\,x^n = q,$$

in which the exponent of x in the first term is twice that in the second. We give a few examples.

1. Find the value of x in the equation $x^4 + 6\,x^2 = 135$. Here the exponent of x in the first term is twice that in the second. The left-hand member may, therefore, be made a perfect square, and the root extracted as in the preceding examples. Performing the operations, we have

$$x^4 + 6\,x^2 + 9 = 144$$
$$x^2 + 3 = \pm\,12$$
$$x^2 = 9$$
$$x = \pm\,3.$$

2. Find the value of x in the equation

$$x - 4\,x^{\frac{1}{2}} = 12.$$

Completing the square,

$$x - 4\,x^{\frac{1}{2}} + 4 = 16.$$

Extracting the root, $x^{\frac{1}{2}} - 2 = \pm 4$;

whence $x^{\frac{1}{2}} = 6$,

and $x = 36$.

3. Find the value of x in the equation

$$(x + 5)^2 + 8(x + 5) = 209.$$

Here, also, the exponent of the quantity enclosed in the parenthesis in the first term is twice the exponent of the same quantity in the second; whence, completing the square,

$$(x + 5)^2 + 8(x + 5) + 16 = 225;$$

extracting the root $(x + 5) + 4 = \pm 15$,

and $x = 6$, or -24.

4. Find the value of x in the equation

$$x^4 - 10x^2 = 375.$$

ANS. $x = 5$.

5. Find the value of x in the equation

$$(x + 12)^{\frac{1}{2}} + (x + 12)^{\frac{1}{4}} = 6.$$

ANS. $x = 4$, or 69

SECTION XII.—INDETERMINATE PROBLEMS.

119. In order that a question may be capable of solution, it should furnish as many distinct equations as there are unknown quantities to be determined. When this is not the case, the question is said to be *indeterminate*.

Let there be the following question:

To find two numbers such that their sum may be equal to 10.

Let x and y be the numbers; then,

$$x + y = 10.$$

Here there are two unknown quantities to be determined, while the question furnishes but one equation. The precise numbers intended in the question are not sufficiently defined, and the question is, therefore, indeterminate.

If we are at liberty to assign, at pleasure, a value to one of the unknown quantities, the question for this value becomes determinate, and the corresponding value of the other unknown quantity may easily be derived from the equation.

Let it, then, be asked, what are all the values of x and y that will satisfy the equation and the enunciation from which it is derived?

If by this question all possible values of x and y, positive or negative, entire or fractional, are meant, there will be no limit, it is evident, to their number, or the answers which may be obtained. But if, as is usually the case, *entire* and *positive* values only are intended, the number of answers will be comparatively few.

From the equation, we have $x = 10 - y$.

Let us now put successively for y the entire and positive numbers between 0 and 10, and deduce the corresponding values of x. We shall thus obtain, it is evident, all the possible solutions of the proposed in entire and positive numbers.

If $\quad y = 1, 2, 3, 4, 5, 6, 7, 8, 9,$

then $\quad x = 9, 8, 7, 6, 5, 4, 3, 2, 1.$

But the four first and the four last of these solutions are evidently the same. The question, therefore, admits of five different solutions, and five only, in entire and positive numbers.

120. That part of algebra which relates to the solution of indeterminate problems is called *Indeterminate Analysis.* It is one of the most interesting branches of the subject, and most fertile in important results. We shall solve a few simple questions, in which the numbers are required to be entire and positive, in order to give some idea of it.

Ex. 1. It is required to divide the number 25 into two parts, one of which may be divisible by 2, and the other by 3.

Let $x =$ the quotient of the part divisible by 2, and $y =$ the quotient of the part divisible by 3; the parts will then be represented by $2x$ and $3y$, respectively, and we have

$$2x + 3y = 25. \quad (1)$$

From which we obtain $\quad 2x = 25 - 3y;$

whence $\quad x = \frac{25 - 3y}{2};$

and, performing the division as far as possible,

$$x = 12 - y + \frac{1 - y}{2};$$

or, changing the signs in the last term so as to make y positive, $\quad x = 12 - y - \frac{y - 1}{2}.$

Here, since, by the question, x and y are entire numbers, and 12 is an entire number, $\frac{y-1}{2}$ must be an entire number, or $y - 1$ must be exactly divisible by 2. Let us put z for the quotient of this division; then

$$y - 1 = 2z;$$

whence $\quad y = 2z + 1, \quad (2)$

and, by substitution, $\quad x = 11 - 3z. \quad (3)$

Now, since x must be *positive* as well as entire, $3z$, it is evident, must not exceed 11, and, by consequence, z can have no value greater than three. We shall obtain, then from the equations (2) and (3), all the values which, from the nature of the question, x and y admit, by assigning to z the entire and positive numbers from 0 to 4, the 0 being included.

If we put $z = 0, \quad z = 1, \quad z = 2, \quad z = 3,$

then $\quad y = 1,\ \ y = 3,\ \ y = 5,\ \ y = 7,$
$\quad\quad\ \ x = 11,\ \ x = 8,\ \ x = 5,\ \ x = 2.$

The question, therefore, admits of 4 different answers, and four only. These are,

1. 22 + 3, 2. 16 + 9, 3. 15 + 10, 4. 21 + 4.

Ex. 2. It is required to pay a debt of 58 francs with pieces of 3 francs and 4 francs, without any other money In how many ways can it be done?

Putting x and y for the pieces, respectively, the question admits of four answers, as follows:

	1st.	2d.	3d.	4th.
$y =$	1,	4,	7,	10,
$x =$	18,	14,	10,	6.

Ex. 3. A sum of 37 dollars was distributed among some poor persons, men and women; each woman received $2, and each man $3. How many men and women were there?

Putting x for the number of women and y for the number of men, the question admits of 6 answers, as follows:

	1st.	2d.	3d.	4th.	5th.	6th.
$y =$	1,	3,	5,	7,	9,	11,
$x =$	17,	14,	11,	8,	5,	2.

Ex. 4. To divide 100 into two such parts that if the first be divided by 5, the remainder will be 2; and if the second be divided by 7, the remainder will be 4.

Let x be the quotient arising from the division of the first part by 5; then the first part will be $5x + 2$. In like manner, the other part will be $7y + 4$, and we shall have

$$5x + 2 + 7y + 4 = 100$$

$$5x + 7y = 94 \quad (1)$$

$$x = 18 - y - \frac{2y - 4}{5}$$

$$x = 18 - y - \frac{2(y - 2)}{5}. \quad (2)$$

Here it is necessary that $2(y-2)$ should be divisible by 5. But, as the factor 2 is not divisible by 5, nor is it a factor of 5, the remaining factor $y-2$ must be divisible by 5. Putting z, as before, equal to the quotient of this division,

we have $$y-2=5z,$$
whence $$x=18-y-2z$$
$$y=5z+2 \quad (3)$$
$$x=16-7z. \quad (4)$$

It is evident that z here must not exceed 2. Putting z successively $=0, 1, 2$, we have

$$\text{for } z=0, \quad z=1, \quad z=2$$
$$y=2, \quad y=7, \quad y=12$$
$$x=16, \quad x=9, \quad x=2.$$

Thus the question admits of three different answers, and of three only, and the parts are

1. $82+18$, 2. $47+53$, 3. $12+88$.

121. In the following examples, the learner will be careful to take out the factor found in the last term of each of the equations corresponding with the equation marked (2) in the example above.

1. Two women have together 100 eggs; one says to the other, When I count my eggs by sevens, there is an overplus of 5. The second replies, When I count mine by tens, there is an overplus of 8. How many had each?

Ans. One had 82 and the other 18;
or, one had 12 and the other 88.

2. In how many ways can a debt of 151 livres be paid, by paying pieces of 11 livres, and receiving in exchange pieces of 7 livres?

Ans. The number of ways is without limit. For the first we have 15 of the one, and 2 of the other.

3. A person bought some sheep and lambs for 253

shillings; the sheep cost 13 and the lambs 5 shillings apiece. How many did he buy of each?

ANS. 1 sheep and 48 lambs, or 6 sheep and 35 lambs, &c.

4. A lady purchases some articles at a store, amounting to $1.07. She has 20 cent pieces only, and the shopkeeper has 3 cent pieces only. In what way can she best pay for her purchase?

ANS. By paying 7 pieces of 20 cents, and receiving in exchange 11 of 3 cents.

5. A gentleman has a piece of work to be done for which he is willing to pay 29 shillings, but is obliged to employ laborers at two different prices, viz., at 5 shillings and 9 shillings a day. In how many ways can he employ laborers at these prices to do the work?

ANS. In one way only.

6. Is it possible to pay a debt of 71 shillings with pieces of 7 shillings and 13 shillings only?

Supposing it possible, let $x =$ the number of pieces of 7 shillings, and $y =$ the number of pieces of 13 shillings; then,

$$7x + 13y = 71, \quad (1)$$

from which we obtain

$$x = 10 - y - \frac{6y - 1}{7}$$

$$x = 10 - y - z$$

$$6y = 7z + 1 \quad (2)$$

$$y = z + \frac{z + 1}{6}$$

$$y = z + z'$$

$$z = 6z' - 1$$

$$y = 7z' - 1 \quad (3)$$

$$x = 12 - 13z'. \quad (4)$$

Here no values, positive or negative, can be given to z' that will render x and y both positive.

It is impossible, therefore, to pay the debt in the manner proposed.

The coefficient of y in equation (2) being greater than unity, an additional auxiliary unknown z' has to be employed. In general, the use of these auxiliary unknown quantities must be continued, until the coefficient of that before the one last employed shall become equal to unity.

7. A company of men and women club together for the payment of a reckoning; each man pays 19 shillings and each woman 16 shillings, and it is found that the women together pay 1 shilling less than the men. How many men and women were there?

Ans. 11 men and 13 women, &c.

8. A merchant purchases some pieces of cloth of two different kinds for $185, giving $11 a piece for the first kind, and $13 a piece for the second. How many pieces of each kind did he purchase?

Ans. 5 of the first and 10 of the second.

9. A coiner has gold of 15 and of 20 carats fine. How many ounces must he take of each, that the mixture may be 18 carats fine?

Let $x =$ the number of ounces of 15, and $y =$ the number of 20 carats fine; then $15x + 20y = 18(x + y)$.

Ans. 2 of 15 and 3 of 20, or in like proportions.

10. A merchant has two kinds of teas; one is worth 37 cents and the other 63 cents a pound. How much must he take of each, that the mixture may be worth 46 cents a pound?

Ans. 17 of the first and 9 of the second, &c.

11. A farmer has wheat at 15 shillings and at 22 shillings a bushel. How much must he take of each to make a mixture worth 17 shillings a bushel?

Ans. 5 of the first and 2 of the second, &c.

The questions solved by the rule of Allígation Alternate in Arithmetic belong to the Indeterminate Analysis. The three last examples are sufficient to show how questions of this kind are performed, when the components of the required quantity are limited to two.

122. We add the following question:

Two bodies set out, at the same time, from a given point in the circumference of a circle, and move upon it with different velocities in the same direction; when will they again be together?

Let v' and v be the velocities, or spaces passed over in the unit of time, by the two bodies, respectively, v' being greater than v. Let the length of the circumference be represented by c, and let x be the time in which the bodies will again be together. Then the spaces passed over by the two bodies in the time x will be $v'x$ and vx, respectively. Now, in order that the two bodies may again be together, it is necessary, and it is sufficient, that the body which moves with the greater velocity should pass over an exact number of circumferences + the space passed over in the same time by the other body. We shall have, therefore, n representing any entire number,

$$v'x = nc + vx,$$

whence

$$x = \frac{nc}{v' - v}.$$

Ex. 1. The hour and minute hands of a watch are together at 12 o'clock. When will they again be together?

Regarding the hour as the unit of time, we shall have $v' = 60$, $v = 5$, and $c = 60$. The formula for x, in this particular case, will be, therefore,

$$x = \frac{n\,60}{55}.$$

Putting n successively $= 1, 2, 3$, &c., we obtain the values of x required.

ANS. In $1\frac{1}{11}$, $2\frac{2}{11}$, $3\frac{3}{11}$, &c., hours.

EX. 2. Two men start together from the same point, and travel round an island 73 miles in circumference. A travels 5 miles and B 3 miles an hour. When will they again be together? ANS. In $36\frac{1}{2}$, 73, $109\frac{1}{2}$, &c., hours.

SECTION XIII.—THEORY OF LOGARITHMS.

123. Various expedients have been devised to abridge the labor of numerical calculations. Among these, the most remarkable is the invention of logarithms.

To give an elementary view of this subject, we write in one line the different powers of the number 2, for example, and in another line, above them, the exponents of these powers. It being recollected, moreover, that 2^0, or that 2 with zero for its exponent, is equivalent to unity, we place unity for the first term in the series of powers, and 0 for the corresponding exponent, thus:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots .$$
$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 \ldots .$$

The first of these two series, it will be perceived, is a progression by difference, the ratio of which is 1, and the second is a progression by quotient, the ratio of which is 2. It will be seen, also, that the addition of the exponents corresponds to the multiplication of the powers or numbers to which they belong, and the subtraction of the exponents to the division of these numbers. Thus, if we add together the exponents 3 and 5, the number 8 which results will be the exponent of 256, the product of 8 by 32, the numbers to which the exponents 3 and 5 belong, respectively. In like manner, if we subtract the exponent 6 from the exponent 10 the number 4 which results is the exponent of 16, or the quotient of 1024, the number corresponding to the exponent 10, by 64, the number corresponding to the exponent 6.

It is easy, then, to see that if the second series, or series of powers, were so enlarged as to comprehend all possible numbers, and the series of exponents were also extended to correspond with them, we might, by aid of the two series, make addition take the place of multiplication, and subtraction that of division.

Such a double series of numbers is called a system of *logarithms*, and, when conveniently arranged in tables, a table of logarithms. The exponents are called the logarithms of the numbers to which they respectively belong; and the invariable number, raised to successive powers, in forming the system, as the 2 in the example above, is called the *base* of the system.

Can we form such a system? Can we, for example, by raising 2 to suitable powers, generate all possible numbers; or, conversely, regarding all numbers as derived from 2, raised to the requisite powers, can we find the exponents of these powers?

The solution of this question gives rise to an equation of a different form from any we have hitherto considered. Thus, to take a particular case, suppose that 7 is considered a power of 2, what is the exponent of this power? The question gives rise, it is evident, to the equation

$$2^x = 7,$$

in which the unknown quantity is an exponent. This is called an *exponential equation*.

We must limit ourselves to the remark that the equation is susceptible of solution, and that the value of x may be determined, either exactly or with such degree of approximation as we please.

Below we have written a table of numbers, from 1 to 30 inclusive, and by the side of them the exponents of the powers to which the number 2 must be raised, in order to produce these numbers.

N.	Log.	N.	Log.	N.	Log.
1	0.0000	11	3.4594	21	4.3922
2	1.0000	12	3.5849	22	4.4594
3	1.5849	13	3.7000	23	4.5235
4	2.0000	14	3.8073	24	4.5849
5	2.3219	15	3.9065	25	4.6438
6	2.5849	16	4.0000	26	4.7000
7	2.8073	17	4.0874	27	4.7548
8	3.0000	18	4.1699	28	4.8073
9	3.1699	19	4.2479	29	4.8577
10	3.3219	20	4.3219	30	4.9065

This is called a table of logarithms; the numbers are written in one column, and the exponents or logarithms, calculated to four places of decimals, are written by their side.

Let it be required to multiply 3 by 9. From the table we find

$$1.5849 = \log.\ 3$$
$$3.1699 = \log.\ 9,$$

and, by addition, $4.7548 = \log.\ 27$, the answer.

Instead of 2, we might have taken for the base of the system any other convenient number, 3, for example. Writing as above a few of the powers of 3, we have

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \ldots$$
$$1,\ 3,\ 9,\ 27,\ 81,\ 243,\ 729,\ 2187,\ 6561,\ 19683 \ldots$$

This system will be found to have the same properties with the first, which has 2 for its base.

Let it be required, for example, to multiply 729 by 27. We have $6 = \log.\ 729$

$3 = \log.\ 27,$

and, by addition, $9 = \log.\ 19683$, the answer.

124. Let a represent generally the base of a system of logarithms, and N any number in the system; we shall have the general equation, $a^x = N$, in which the value of x is to be determined for all possible values of N.

In this equation, x is the logarithm of N. And, in general, *the logarithm of a number is the exponent of the power to which it is necessary to raise an invariable number taken as a base, in order to produce the number.*

Supposing a table of logarithms formed by means of this equation, we will now demonstrate the properties which these logarithms possess.

Let us designate by $N, N', N'' \ldots$ any numbers whatever and by $x, x', x'' \ldots$ their logarithms; we shall have the series of equations

$$a^x = N,\ a^{x'} = N',\ a^{x''} = N'' \ldots$$

Multiplying next these equations member by member,

$$a^x \times a^{x'} \times a^{x''} \ldots = N \times N' \times N'' \ldots,$$

or

$$a^{x + x' + x''} = N N' N''.$$

Now, for the same reason that x is the logarithm of N, $x + x' + x''$ will be the logarithm of $N N' N''$. In all cases, therefore, we have this principle:

The logarithm of a product is equal to the sum of the logarithms of the factors of this product.

A multiplication, then, being proposed, if we take from the table the logarithms of the numbers to be multiplied, the sum of these will be the logarithm of the product sought. Seeking next this logarithm in the table, the number correspond ing to it will be the product sought.

Let it be required next to divide the number N by the number N', and, as before, let x, x', be the logarithms of these numbers; then we have the equations

$$a^x = N,\ a^{x'} = N';$$

dividing these member by member,

$$\frac{a^x}{a^{x'}} = \frac{N}{N'},\ \text{or}\ a^{x-x'} = \frac{N}{N'};$$

whence $\log. \frac{N}{N'} = x - x' = \log. N - \log. N'$.

Hence, *the logarithm of a quotient is equal to the difference between the logarithm of the divisor and that of the dividend.*

If, then, it be proposed to divide one number by another, from the logarithm of the dividend we subtract the logarithm of the divisor; the remainder will be the logarithm of the quotient; seeking, therefore, this logarithm in the table, the number corresponding will be the quotient sought.

From what has been done, it will be seen that, in the use of a table of logarithms, *addition may be made to take the place of multiplication, and subtraction that of division.*

Let it next be required to raise the number N to a power denoted by m. We have the equation

$$a^x = N,$$

or, raising both members to the mth power,

$$a^{mx} = N^m;$$

whence the logarithm of $N^m = mx = m \log. N$.

That is, *the logarithm of any power of a number is equal to the product of the logarithm of this number by the exponent of the power.*

To raise a number, therefore, to a power, by means of logarithms, we multiply the logarithm of the proposed by the exponent of the power to which it is to be raised; the number in the table corresponding to this product will be the power sought.

Again, let it be required to find the mth root of a number N. We have, as before,

$$a^x = N.$$

Taking the mth root of both members,

$$a^{\frac{x}{m}} = N^{\frac{1}{m}}; \text{ whence } \log. N^{\frac{1}{m}} = \frac{x}{m} = \frac{\log. N}{m}.$$

That is, *the logarithm of the root of any degree of a number is equal to the logarithm of the number divided by the index of the root.*

125. The properties above are altogether independent of the base; we may take, therefore, as we have before said, any number we please, with the exception of unity, for the base of a system of logarithms.

The number 10, however, is found in practice to be the most convenient base. To calculate this system, we have to calculate the value of x in the equation $10^x = N$, for all the numbers we wish to place in the system.

Taking a few of the first powers, we have

$$x = 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5 \ldots.$$
$$N = 1, 10, 100, 1000, 10000, 100000 \ldots.$$

From this it will be seen that in the system, the base of which is 10,

1°. The logarithms of numbers between 1 and 10 will be a fraction.

2°. The logarithms of numbers between 10 and 100 will be 1 and a fraction; between 100 and 1000, 2 and a fraction, and so on.

The logarithm of 3, for example, calculated in this system, is 0.477121; that of 15 is 1.76091; that of 120 is 2.079181, and so on.

In general, it will be seen that the figure in the entire part of the logarithm is one less than the number of figures in the number. Thus, if the number consists of two figures, the number in the entire part of the logarithm will be 1 less than 2, or unity; if the number consists of three figures, it will be 1 less than 3, or 2, and so on.

This entire part is called the *characteristic* of the logarithm. And, as it is easy, from the principle above, to determine the entire part from the number, and the number of figures in the number from the entire part of the logarithm, it is usual to omit the characteristics in the table, to save the room.

Since the logarithm of 10 is 1, that of 100 is 2, and so on, it follows that the logarithm of a number multiplied by 10, 100, . . . will be 1, 2, . . . units greater than the logarithm of this number. Thus, the logarithm of 5 being 0.698970, the logarithm of 10 times 5, or 50, is 1.698970, that of 100 times 5, or 500, is 2.698970, and so on. The decimal part of the logarithm will, therefore, be the same for this number, or for its product or quotient by 10, 100, and so on. This is one of the principal advantages of the system of logarithms, the base of which is 10; since we have frequent occasion to multiply or divide numbers by 10, 100, and so on, operations reduced in this case to the simple addition or subtraction of units.

126. It is usual to accompany tables of logarithms with sufficient explanations of the mode of using them. We add a single explanation here.

Let it be proposed to find, for example, the logarithm of $\frac{75}{100}$. The logarithm of 75 is 1.875061; that of 100 is 2.000000. Subtracting the latter from the former, we have

$-124939 = \text{log.}\ \frac{75}{100}$. But this logarithm, which is altogether negative, is inconvenient in practice, and a more convenient one is to be sought.

The fraction $\frac{75}{100}$ returns to $\frac{1}{100} \times 75$; whence

log. $.75 = \text{log.}\ \frac{1}{100} + \text{log.}\ 75 = -2 + 1.875061 = -1 + .875061$; or, placing the sign $-$ over the 1, to show that the characteristic only is negative, $\bar{1}.875061$.

This, it will be perceived, is only a continuation of the principle already enunciated, according to which the logarithm of a number 10, 100, ... times less than the proposed number, is found by subtracting 1, 2, ... units from the characteristic.

Thus the logarithm of $750 = 2.875061$; whence

$$\begin{aligned} \text{log.}\ 75 &= 1.875061 \\ \text{log.}\ 7.5 &= 0.875061 \\ \text{log.}\ .75 &= \bar{1}.875061 \\ \text{log.}\ .075 &= \bar{2}.875061 \\ \text{log.}\ .0075 &= \bar{3}.875061 \end{aligned}$$

In general, the negative characteristic of the logarithm of a decimal fraction has for its numerical value a number of units equal to the number which marks the place of the first significant figure of the fraction from the place of units.

Supposing the learner furnished with a table of logarithms, we will now give some examples of their use. In the tables we employ, the logarithms are calculated to the sixth decimal place inclusive.

1. Let it be required to multiply 863 by 429.

$$\begin{aligned} \text{log.}\ 863 &= 2.936011 \\ \text{log.}\ 429 &= 2.632457 \\ \hline & \quad 5.568468. \end{aligned}$$

This logarithm is not found in the tables. It is comprised however, between 5.568436, the logarithm of 370200, and 5.568554, the logarithm of 370300. The difference between these two logarithms is 118; the difference between the less of these logarithms and the proposed is 32. We shall have, therefore, the following proportion,

118 : 100 :: 32 : 27, nearly.

Adding 27 to 370200, the number corresponding to the less logarithm, we have 370227 for the answer.

2. Divide 173052 by 253.

The numbers in the tables extending only to 10000, the logarithm of the dividend is not contained in them. It can, however, be easily found. The nearest number to it, contained in the tables, is 1730. We regard, then, for the moment, the proposed as 1730.52. The logarithm of this last is comprised between the logarithms of 1730 and 1731 The difference between these two logarithms is 251; $\frac{52}{100}$ therefore of this difference, added to the logarithm of 1730, will give the logarithm of 1730.52, nearly; thus, log. 1730.52 = 3.238177; whence, adding two units to this last to obtain the logarithm of the proposed, we have

$$\begin{array}{rl} \text{log. } 173052 = & 5.238177 \\ \text{log. } 253 = & 2.403121 \\ \hline \text{log. } 684, \text{ Ans.} & 2.835056. \end{array}$$

3. Multiply .735 by .0087.

$$\begin{array}{rl} \text{log. } .735 = & \bar{1}.866287 \\ \text{log. } .0087 = & \bar{3}.939519 \\ \hline \text{log. } .0063945, \text{ Ans.} & \bar{3}.805806. \end{array}$$

4. Divide .053 by 797.

$$\begin{aligned} \text{log. } .053 &= \bar{2}.724726 = \bar{3} + 1.724276 \\ \text{log. } 797 &= \qquad\qquad\qquad\quad 2.901458 \\ \hline \text{log. } .0000665, \text{ Ans.} & \qquad\qquad\qquad\quad \bar{5}.822818. \end{aligned}$$

To render the subtraction required in this example possible, we change the characteristic $\bar{2}$ into $\bar{3}+1$, which has the same value; this furnishes a ten to be joined with 7 for the subtraction of 9, the left-hand figure of the decimal part. A similar preparation, it is evident, must be made in all cases of the kind.

5. To find the 5th power of .125.

$$\begin{aligned} \text{log. } .125 &= \bar{1}.096910 \\ & \qquad\qquad 5 \\ \hline \text{log. } .000030519, \text{ Ans., nearly. } & \bar{5}.484550. \end{aligned}$$

6. To find the third root of .0375.

The logarithm of .0375 is $\bar{2}.574031$. The negative characteristic, $\bar{2}$, of this logarithm, is not divisible by 3, the index of the root required; neither can it be joined to the positive part, on account of the different sign. But, if we add $-1+1$ to the characteristic, which will not alter its value, it becomes $\bar{3}+1$; the negative part is then divisible by 3, and the 1, being positive, may be joined to the fractional part.

Thus log. $.0375 = \bar{2}.574031 = \bar{3} + 1.574031$; whence, dividing by 3, we have $\bar{1}.524677 =$ log. .334716, the root required.

SECTION XIV.—Compound Interest.

127. One of the most important applications of logarithms is to questions upon the interest of money.

Interest is of two kinds, *simple* and *compound*. If interest is paid upon the principal only, it is called, as we have seen, *simple interest*; but, if the interest, as it becomes due, is added to the principal, and interest is paid upon the whole, it is then called *compound interest*.

We have already investigated rules for simple interest; we will now do the same for compound interest. The general problem may be stated as follows:

To determine what sum a given principal p will amount to, in a number n of years, at a given rate r, at compound interest.

The amount of unity for 1 year will be $1 + r$; that of p units, therefore, will be $p(1 + r)$.

For the second year, $p(1 + r)$ will be the principal, and its amount will be $p(1 + r)(1 + r)$; or $p(1 + r)^2$.

In like manner, the amount at the end of the third year will be $p(1 + r)^3$, at the end of the fourth, $p(1 + r)^4$, and so on.

Let us put A for the amount at the end of the n years then $$A = p(1 + r)^n. \quad (1)$$

This is a general formula for compound interest, according to which, to find the amount, *we multiply the principal by the rate increased by unity, and raised to a power denoted by the number of years.*

These operations, especially that of finding the requisite power when the number of years is large, will be much facilitated by the use of logarithms.

If we take the logarithms of both sides of the formula (1) we have, $\log. A = \log. p + n \log. (1 + r)$.

Ex. 1. What will be the amount of \$5000 in 40 years at 4 per cent., compound interest?

By the formula, $\log. A = \log. 5000 + 40 \log. 1.04$.

Ans. \$24007.90.

Ex. 2. What will be the amount of $375 in 10 years, at 6 per cent., compound interest? Ans. $671.57.

Ex. 3. What will be the amount of $763 in 6 years, at 5 per cent., compound interest? Ans. $1022.49.

128. The equation $A = p(1 + r)^n$ contains four quantities, A, p, r and n, any one of which may be determined when the others are known. It gives rise, therefore, to four different questions.

1°. To determine A, when p, r, and n, are given; or, *the principal, rate and number of years being given, to find the amount.*

This question we have already solved.

2°. To determine p, when A, r, and n are given; or, to find *what principal, put at compound interest, will amount to a given sum, in a certain number of years, at a given rate.*

Resolving the general equation with reference to p, we have

$$p = \frac{A}{(1+r)^n};$$

or, by logarithms, $\log. p = \log A - n \log. (1 + r)$.

Ex. 1. What principal will amount to $350.95 in 4 years, at 4 per cent., compound interest? Ans. $300.

Ex. 2. How much money must be placed out at compound interest to amount to $1000 in 20 years, the interest being 5 per cent.? Ans. $376.89.

3°. To determine r, when A, p, and n are known; or, *to find at what rate a given sum must be put at compound interest, in order to amount to another given sum in a given time.*

Resolving the general equation with reference to r, we have

$$(1 + r) = \sqrt[n]{\frac{A}{p}};$$

or, by logarithms,

$$\log. (1 + r) = \frac{\log. A - \log. p}{n}.$$

Having, by means of this last, determined the value of $1 + r$, that of r will be easily found.

Ex. 1. A capital of \$3200, having been at compound interest for 80 years, has amounted to \$34050.84; at what rate per cent. was it put out? ANS. 3 per cent.

Ex. 2. A sum of \$1350, having been at compound interest for 3 years, has amounted to \$3562.79; at what rate per cent. was it put out? ANS. 5 per cent.

4°. To determine n, when A, p, and r are given; or, *to find for what time a given sum must be put at compound interest, to reach, at a given rate, a given amount.*

Making n the unknown quantity in the general formula, we obtain

$$n = \frac{\log. A - \log. p}{\log. (1 + r)}.$$

Ex. 1. In what time will £500 amount to £900, at 5 per cent.? ANS. 12.04 years.

Ex. 2. In what time will \$200 amount to \$238.20, at 6 per cent.? ANS. In 3 years.

129. We close with the following question: In what time will a sum be doubled, tripled, &c., at compound interest?

To prepare a formula for this, let k denote 1, 2, 3 . . . then in the general formula we shall have $A = k p$, and we shall have

$$k p = p (1 + r)^n;$$

whence

$$n = \frac{\log. k}{\log. (1 + r)}.$$

Here n, it is evident, is independent of p; that is, whatever the sum put out, it will be doubled, tripled, &c., in the same time.

Ex. In what time will a sum be doubled, at 6 per cent. compound interest? ANS. In 11.895 years, nearly.

SECTION XV.—PRAXIS.

130. The following examples may be employed as additional exercises in the articles to which they are referred:

ART. 2. Let the learner express in common language the reasonings necessary for the solution of the following questions:

1. A farmer gave his laborers $96, paying each man $6, and each boy $2. There were as many boys as men. How many were there of each?

2. A boy bought as many pen-holders as writing-books, paying 3 cents for a pen-holder, and 6 cents for a writing-book; he expended in all 27 cents. How many of each did he buy?

3. In a certain school there are 30 pupils in two classes, and the grammar class is twice as large as the reading class. How many pupils were there in each class?

4. A man sold a knife for 50 cents, by which he gained four times the cost. How much did it cost?

ART. 5. Exercises to follow question 33:

1. The sum of two numbers is 36, and one is half as large as the other. What are the numbers?

2. In an orchard of 54 trees, there are one-third as many apple as pear trees. How many are there of each?

3. A and B enter into partnership, and gain $105. A put in a certain sum, and B three-fourths as much. What is each man's share of the gain?

4. A person being asked his age, replied, "If from half my age you subtract one-fifth, the remainder will be 18." What was his age?

ART. 9. Exercises to follow question 5:

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1. The sum of two numbers is 68, and the second is 7 less than twice the first. What are the numbers?

2. The sum of three numbers is 56; the second is 9 less than three times the first, and the third is 5 more than twice the first. What are the numbers?

3. If 8 be added to twice a certain number, and 12 be subtracted from three times this number, the sum and difference will be equal. What is the number?

4. Two persons talking of their ages, the first says to the second, "My age is not so much by 14 years as three times yours, and if twice your age be subtracted from 100, the remainder will be equal to our joint ages." What was the age of each?

5. A sum of money is to be divided between two persons, A and B, so that B shall have $10 more than twice as much as A. Now, if $90 be added to twice A's share, the sum will be equal to both their shares together. What was the share of each?

6. In a certain school, the second class contains four times as many pupils, wanting 5, as there are in the first class, and if the number in the third class, which is twice that of the first, be subtracted from 100, the remainder will be equal to the number in the other two classes together. How many were there in each class?

ART. 11. Additional questions:

1. A woman carrying her eggs to market broke one-fifth on the way; while there, she sold three-tenths of her whole number, and still had 20 left. How many had she at first?

2. In a college one-fifth of the students are Seniors, one-fourth Juniors, three-tenths Sophomores, and there are 40 in the Freshman class. What is the whole number of students?

3. A man uses one-third of his farm for pasture, one-sixth is meadow, one-twelfth is woodland, and the remaining 20 acres he cultivates with the plough. How many acres has he in his farm?

4. A shepherd has his sheep in two pastures. In the first there are two-thirds of his flock, wanting 20, and in the second, three-fourths, wanting 30, and there is the same number in each pasture. How many sheep has he?

5. Out of a cask of wine which had leaked away a third part, 21 gallons were drawn, and the cask being then gauged, was found to be half full. How much did it hold?

Art. 12. Exercises to follow question 21:

1. The sum of two numbers is 20, and twice the greater is equal to three times the less. What are the numbers?

2. A farmer has 96 sheep in two pastures, and three times the number of sheep in the larger pasture will be equal to five times the sheep in the other. How many sheep are there in each pasture?

3. Two men together spend \$120. Five times the money A spends is equal to seven times the money B spends. How many dollars does each spend?

4. Find a number such that if 7 be added to it, and again 5 be subtracted from it, three times the sum will equal four times the difference.

5. Two men commence trade, each with the same sum; A gains \$50, B loses \$40, when it is found that five times A's money is equal to seven times B's. What had each at first?

2. Exercises to follow those under question 22:

What will be the expressions for the remainder when the following subtractions are performed:

1. $x + 1$ subtracted from $3x$.

2. $21 + 9x$ subtracted from $11x$.
3. $3x + 15$ subtracted from 20.
4. $x - 15$ subtracted from x.
5. $7x - 60$ subtracted from $8x - 30$.
6. $9x + 100$ subtracted from $10x - 190$.
7. 3 times $x - 1$ subtracted from $7x$.
8. 7 times $x + 5$ subtracted from $13x - 20$.
9. 10 times $75 - 8x$ subtracted from $1240 - 4x$.
10. 6 times $90 - 3x$ subtracted from $720 - 5x$.
11. $3(4x - 20)$ subtracted from $(3x - 20)$.
12. $5(3x - 25)$ subtracted from $7(9x + 15)$.

MISCELLANEOUS EQUATIONS OF THE FIRST DEGREE.

131. Find the values of x and y in the following equations:

1. $\frac{x-5}{4} + 6x = \frac{284-x}{5}$. Ans. $x = 9$.

2. $x + \frac{11-x}{3} = \frac{19-x}{2}$. Ans. $x = 5$.

3. $\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$.

Ans. $x = 6$.

4. $(x-24)^{\frac{1}{2}} = x^{\frac{1}{2}} - 2$.

Squaring both sides,

$$x - 24 = x - 4x^{\frac{1}{2}} + 4;$$

whence $x^{\frac{1}{2}} = 7$, and $x = 49$.

5. $(4x + 12)^{\frac{1}{2}} = 16$. Ans. $x = 61$.

6. $(12 + x)^{\frac{1}{2}} = 2 + x^{\frac{1}{2}}$. Ans. $x. = 4$.

7. $(4x + 21)^{\frac{1}{2}} = 2x^{\frac{1}{2}} + 1$ Ans. $x = 25$.

8. $(x-32)^{\frac{1}{2}} = 16 - x^{\frac{1}{2}}$. Ans. $x = 81$.

9. $12 - x : \frac{x}{2} :: 4 : 1$. Ans. $x = 4$.

10. $\frac{10+x}{5} : \frac{4x-9}{7} :: 14 : 5$. Ans. $x = 4$.

11. $\frac{17-4x}{4} : \frac{15+2x}{3} - 2x :: 5 : 4$.

Ans. $x = 3$.

12. $\left.\begin{array}{l} x+1 : y :: 5 : 3 \\ \frac{2x}{3} - \frac{5-y}{2} = \frac{41}{12} - \frac{2x-1}{4}. \end{array}\right\}$

Ans. $x = 4$, $y = 3$.

13. $\left.\begin{array}{l} \frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}. \\ \frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4}. \end{array}\right\}$

Ans. $x = 7$, $y = 10$.

MISCELLANEOUS EQUATIONS OF THE SECOND DEGREE.

132. Find the values of x and y in the following equations:

1. $\left.\begin{array}{r} x+y = 10 \\ xy = 21. \end{array}\right\}$ Ans. $x = 7$, $y = 3$.

2. $\left.\begin{array}{r} x-y = 4 \\ xy = 45. \end{array}\right\}$ Ans. $x = 9$, $y = 5$.

3. $\left.\begin{array}{r} 5x+3y = 66 \\ xy = 63. \end{array}\right\}$ Ans. $x = 9$, $y = 7$.

4. $\left.\begin{array}{r} x+y : x :: 7 : 5 \\ xy + y^2 = 126. \end{array}\right\}$ Ans. $x = \pm 15$, $y = \pm 6$.

5. $\left.\begin{array}{r} x^{\frac{1}{2}} + y^{\frac{1}{2}} : x^{\frac{1}{2}} - y^{\frac{1}{2}} :: 4 : 1 \\ xy = 225. \end{array}\right\}$ Ans. $x = \pm 25$, $y = \pm 9$.

6. $\left.\begin{array}{r} x^2 + xy = 12 \\ y^2 + xy = 24. \end{array}\right\}$

Adding the two equations,

$$x^2 + 2\,x\,y + y^2 = 36.$$

Extracting the square root of both sides,

$$x + y = \pm\, 6.$$

Substituting in the first equation 6 for $x + y$,

$$6\,x = 12;$$

whence $$x = 2 \text{ and } y = 4.$$

7. $\left.\begin{array}{l} x^2 - x\,y = 21 \\ x\,y - y^2 = 12. \end{array}\right\}$ Ans. $x = 7,$ $y = 4.$

8. $\left.\begin{array}{l} x^2 + x\,y = 60 \\ x\,y + y^2 = 84. \end{array}\right\}$ Ans. $x = 5,$ $y = 7.$

9. $\dfrac{x}{x+60} = \dfrac{7}{3\,x - 5}.$ Ans. $x = 14$, or $-10.$

10. $\dfrac{10 - 3\,x}{x} = \dfrac{4\,x}{10 - 3\,x}.$ Ans. $x = 2$, or $11\frac{19}{21}.$

11. $x + 5 - (x + 5)^{\frac{1}{2}} = 6.$ Ans. $x = 4$, or $-1.$

12. $x + 16 - 7\,(x + 16)^{\frac{1}{2}} = 10 - 4\,(x + 16)^{\frac{1}{2}}.$
Ans. $x = 9$, or $-12.$

MISCELLANEOUS QUESTIONS.

133. I. *Questions producing Equations of the First Degree.*

1. A person bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew four gallons, and then found that there were four times as many gallons remaining in the larger as in the other. How many were there in each at first?

Ans. 36 and 12 gallons respectively.

2. A merchant has wines at 9 shillings and at 13 shillings a gallon, and he wishes to make a mixture of 100 gallons, that shall be worth 12 shillings a gallon. How many gallons of each must he take?

Ans. 25 gallons at 9*s*. and 75 at 13*s*.

3. A person at play won twice as much money as he began with, and then lost 16 shillings. After this, he lost four-fifths of what remained, and then won as much as he began with; and, counting his money, found he had 80 shillings. What sum did he begin with?

Ans. 52 shillings.

4. Two boys standing on opposite banks of a river, the breadth of which they wished to ascertain, A first shot an arrow across the river, and it flew 13 yards beyond the bank on which B stood. B then took it up, and from the place where it had fallen, shot it back across the river; it now fell $9\frac{2}{7}$ yards beyond the bank where A stood. They afterwards found that 8 times the distance A shot, and 7 times the distance B shot, would, together, be just equal to one mile. What was the breadth of the river?

Ans. 100 yards.

5. A courier passing through a certain place A, travels at the rate of 13 miles in 2 hours; 12 hours afterward, another passes through the same place, travelling the same road, at the rate of 26 miles in 3 hours. How long and how far must he travel before he overtakes the first?

Ans. 36 hours and 312 miles.

6. Two persons, A and B, agree to purchase a house together, worth $1200. Says A to B, Give me two-thirds of your money and I can purchase it alone; but says B to A, If you give me three-fourths of your money, I shall be able to purchase it alone. How much had each?

Ans. A $800, B $600.

7. From a company of ladies and gentlemen, 15 ladies retire; there are then left 2 gentlemen to each lady. After which, 45 gentlemen depart, when there are left 5 ladies to each gentleman. How many were there of each at first?

Ans. 50 gentlemen and 40 ladies.

8. A and B speculate with different sums. A gains £150, B loses £50; and now A's stock is to B's as 3 to 2. But had A lost £50, and B gained £100; then A's stock would have been to B's as 5 to 9. What was the stock of each? ANS. A's £300, B's £350.

9. A father divides his property between his two sons. At the end of the first year, the elder had spent one-quarter of his, and the younger had made $1000, and their property was then equal. After this, the elder spends $500, and the younger makes $2000, when it appears the younger has just double the elder. What had each from the father?

ANS. The elder $4000, the younger $2000.

10. Two pieces of cloth of equal goodness, but of different lengths, were bought, the one for £5, the other for £6 10*s*. Now, if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece, and how much did they cost a yard? ANS. 20 and 26 yards, and the price is 5*s*.

11. Three persons, A, B, and C, together possess $3640. If B gives A $400 of his money, then A will have $320 more than B; but if B takes $140 of C's money, then B and C will have equal sums. How much has each?

ANS. A $800, B $1280, C $1560.

12. A number is expressed by three figures. The sum of these figures is 11; the figure in the place of units is double that in the place of hundreds; and when 297 is added to this number, the order of the figures will be reversed. What is the number? ANS. 326.

II. *Questions producing Equations of the Second Degree.*

1. A gentleman had occasion to hire 16 boys and 21 girls. Each girl received four-fifths as much as a boy. The number of days they worked was equal to one-tenth of a boy's

daily wages. At the end of the term, the gentlemen paid them $82. Required their wages, and the number of days they worked.

ANS. Boys, 50 cts., girls, 40 cts. And they worked 5 days.

2. A and B carried 100 eggs between them to market, and each received the same sum. If A had carried as many as B, he would have received 18 pence for them; and if B had taken only as many as A, he would have received only 8 pence. How many nad each?

ANS. A had 40, and B 60.

3. Two partners, A and B, gained $60 in trade: when, dividing their gain, B took $20 for his share. A's money continued in trade 4 months, and if the number 50 be divided by A's money, the quotient will give the number of months that B's money, which was $100, continued in trade. What was A's money, and how long did B's money continue in trade?

ANS. A's money was $50, and B's money was one month in trade.

4. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number.

ANS. 23.

5. In a right-angled triangle the hypothenuse or longest side is 10 feet, and the area is 24 square feet. What is the length of the other two sides?

NOTE. In solving this and other similar questions, it will be recollected that the square of the hypothenuse is equal to the sum of the squares of the other two sides, and the area is equal to one-half the product of these sides.

ANS. The sides are 8 and 6 feet, respectively.

6. A flag-staff erected on level ground is 117 feet in height. A rope, 125 feet long, is extended from the top of it to the ground. At what distance from the foot of the staff does the rope reach the ground? Ans. $\pm$ 44 feet.

How shall we interpret the negative answer to this question?

If we suppose the rope to be 100 feet in length, an imaginary result is obtained. What absurdity in the question leads to this result?

7. The captain of a privateer, descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her; and then observing the trader steering in a direction perpendicular to her former course, changed his own course, so as to overtake her without making another tack. On comparing their reckonings, it was found that the privateer had run at the rate of 10 miles an hour, and the trading vessel at the rate of 8 hours in the same time. Required the distance sailed by the privateer. Ans. 25 miles.

8. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares? Ans. 21 and 25.

9. Two retailers jointly invested $500 in business, to which each contributed a certain sum; the one let his money remain 5 months, the other only 2, and each received back $450 capital and profit. How much did each advance?

Ans. One $200, the other $300.

10. A merchant bought some linen and muslin for $10.50, the whole number of yards being 50; and each cost as many cents per yard as there were yards of the other. How much of each did he purchase?

Ans. 35 or 15 of linen, and 15 or 35 of muslin.

11. Two drapers cut each of them a certain number of yards from a piece of cloth; one, however, 3 yards less than the other; and jointly receive for them $50. "At my own price," said the first to the other, "I should have received $25 for your cloth." "I must admit," answered the other, "that at my low price I should have received for your cloth no more than $16." How many yards did each sell?

ANS. The one 8, the other 5 yards.

12. Two persons, A and B, jointly invested $2000 in business. A let his money remain 17 months, and received back in capital and profit $1710; the other allowed his money to remain 12 months, and received in capital and gain $1040. What was each partner's stock in trade?

ANS. One $1200, the other $800.

13. A person dies, leaving children and a fortune of $46,800, which, by the will, is to be divided equally amongst them. It happens, however, that immediately after the death of the father, two of his children also die. If, in consequence of this, each remaining child receives $1950 more than otherwise entitled to by the will, how many children were there? ANS. 8.

14. A man had a field whose length exceeded its breadth by 5 rods. He gave 3 dollars a rod to have it fenced; and the whole number of dollars was equal to the number of square rods in the field. Required the length and breadth of the field. ANS. 15 and 10 feet respectively.

15. An uncle said to his nephew, I have in my mind a certain number of shillings, which I will give you, if you guess how many there are. Four times the number diminished by its square is equal to 5. What is the number?

ANS. The problem is impossible.

16. To divide the number 100 into two such parts that the sum of their square roots may be 14.

ANS. 64 and 36.

17. A square court-yard has a rectangular gravel-walk around it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164. Required the area of the court. ANS. 256 yards.

18. A body of men were formed into a hollow square, 3 deep, when it was observed that, with the addition of 25 men to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square. ANS. 936.

19. The difference between the hypothenuse and base of a right-angled triangle is 6, and the difference between the hypothenuse and the perpendicular is 3. What are the sides? ANS. 15, 12, and 9.

III. *Questions in Progressions.*

1. A person bought 7 books, the particular prices of which are in arithmetical progression. The price of the next above the cheapest was 8 shillings, and the price of the dearest, 23 shillings. What was the price of each book?

Let $x =$ the price of the cheapest, $y =$ the common difference. The equations of the question will then be $x + y = 8$, $x + 6y = 23$. ANS. 5, 8, 11, &c., shillings.

2. There are three numbers in arithmetical progression. Their sum is equal to 18, and the product of the two extremes, added to 25, is equal to the square of the mean. What are the numbers?

Let $y =$ the common difference, and let $x - y, x, x + y$, be the numbers respectively. Ans. 1, 6, and 11.

3. A number consists of three digits, which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. Required the number. Ans. 234.

4. There are four numbers in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is equal to 32. What are the numbers?

Let $2\,y =$ the common difference, and let the numbers be $x + 3\,y, x + y, x - y, x - 3\,y$, respectively. Then, by the first condition, $4\,x^2 + 20\,y^2 = 276$;
by the second, $4\,x = 32$.

Ans. The numbers are 11, 9, 7, and 5.

5. There are four numbers in arithmetical progression, the product of the extremes being 22, and that of the means 40. What are the numbers? Ans. 2, 5, 8, and 11.

6. After A, who travelled at the rate of 4 miles an hour, had been set out two hours and three-quarters, B set out to overtake him, and in order thereto, went four miles and a half the first hour, four and three-quarters the second, and so on, gaining a quarter of a mile every hour. In how many hours would he overtake A?

Let $x =$ the number of hours. By the formulas for progression by difference, Art. 74, we obtain, for the whole number of miles B travelled,

$$\left(9 + (x - 1)\frac{1}{4}\right) \times \frac{x}{2};$$

whence, by the question,

$$\left(9 + (x - 1)\frac{1}{4}\right)\frac{x}{2} = 11 + 4\,x,$$

which, being solved, gives 8 hours for the answer.

7. A and B set out from London at the same time, to go round the world, a distance in the parallel of London of 23661 miles, one going east, the other west. A goes 1 mile the first day, 2 the second, and so on. B goes 20 miles a day. In how many days will they meet, and how many miles will be travelled by each?

ANS. 198 days; A goes 19701, and B 3960 miles.

8. A sets out from a certain place, and travels 1 mile the first day, 2 the second, 3 the third, and so on. In five days afterwards, B sets out, and travels 12 miles a day. How long will A travel before he is overtaken by B?

ANS. 8 or 15 days.

The question leads to a quadratic equation, in which the roots, or values of the unknown quantity, are both positive. If the distances A travels each day are set off in succession on one side of a line, and, beginning at the same point, those which B travels are set off on the other side, it will be seen, from the figure, that A is in advance of B until the end of the 8th day, when B overtakes and passes him. After the 12th day, A gains upon B, and overtakes and passes him on the 15th, after which he is continually in advance of him.

9. A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the third, and so on. After he had been gone three days, a second traveller sets out, and goes 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first?

ANS. In 2 or 9 days.

10. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

Let x, $x\,y$, $x\,y^2$, $x\,y^3$, be the numbers, x being the first, and y the common ratio. We shall have, by the question,

$$x + x\,y = 15$$

$$x\,y^2 + x\,y^3, \text{ or } (x + x\,y)\,y^2 = 60.$$

ANS. 5, 10, 20, and 40.

11. Find three numbers in geometrical progression, whose sum shall be 52, and the sum of the extremes shall be to the mean as 10 to 3. ANS. 4, 12, and 36.

IV. *Indeterminate Problems.*

1. In a foundery two kinds of cannon are cast; each cannon of the first sort weighs 1500 lbs., and each of the second 1900 lbs.; and yet for the second there are used 100 lbs. of metal less than for the first. How many cannons are there of each kind?

ANS. 14 of the first, and 11 of the second, &c.

2. A person purchased some hares and sheep. Each hare cost him 8 and each sheep 27 shillings. He found that he had paid for the hares 97 shillings more than for the sheep. How many hares did he purchase, and how many sheep? ANS. 29 and 5, or, &c.

3. With two measuring rods of different lengths, the one 5 feet and the other 7, it is required to make, by placing them the one after the other, a length of 23 feet.

ANS. The problem is impossible.

4. A boy plays with nuts, and wishes to make small heaps of them, If he puts 13 in each heap, he has 9 over; but if he puts 17 in each, he has 14 over. How many nuts are there? ANS. 269, or, &c.

5. A country woman brings a number of eggs to market, more than 100, but less than 200. She is undetermined whether to sell them in fifteens or by the dozen; for in the first case she would have 4 and in the second 10 over How many eggs had she? ANS. 154.

Is this problem really indeterminate?

6. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis d'ors, valued at 17*s*. each. How must I acquit myself of the debt?

Ans. I must pay him 13 guineas, and he must give me 16 louis d'ors.

7. Is it possible to pay £50 by means of guineas and three shilling pieces only? Ans. Impossible.

8. What is the least entire value of x in the equation $11x + 35y = 497$? Ans. 42.

9. Two bodies move upon the circumference of a circle, with velocities equal to v' and v, respectively. They are now at a distance d apart; when will they again be together?

Let c be the length of the circumference, n any entire number, and x the time required. We shall have, for the general formula,

$$x = \frac{nc + d}{v' - v}.$$

What formulas will express the times when the bodies are together for the first, second, third, &c., times?

10. If the sun moves every day 1 degree, and the moon 13, and the sun is now 60 degrees in advance of the moon, when will they be in conjunction for the first time, second time, and so on?

This problem, it is evident, is only a particular case of the preceding. Ans. In 5 days, 35 days, &c.